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Option Theory

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1 Introduction

In the past decades, financial mathematics and especially stochastic calculus have played an important role in the spectacular development of the modern theory of finance and, particularly, in option theory.

Options or option contracts are derivative securities since their value depends on other “more basic” financial assets. For instance, a European call option on a certain stock gives its holder the right to buy one share of the underlying stock for a fixed price at a certain future time point. The basic problem of option theory is to derive the “fair” price of the option, that is to answer the following question: “How much would you be willing to pay for such a contract?” The famous papers [Black and Scholes 73] and [Cox, Ross and Rubinstein 79] gave first answers to this question, which were suitable to be used in practice as well.

The main aim in these notes is to give a survey in which option contracts and (generally) contingent claims are described, option pricing formulas are derived in a fairly general discrete time mathematical model and so to get an overview by using a unified terminology based on [Harrison and Pliska 81], [Shiryaev, Kabanov, Kramkov, Mel’nikov I 94] and [Shiryaev, Kabanov, Kramkov, Mel’nikov II 94].

For this purpose, we study the notion of discrete time \((B, S)\) security markets (Sections 3-5, 7-8) with two available financial assets: one is labelled bond which is riskless, whereas the other is called stock which is risky. We will describe several important notions such as self-financing strategy, hedging strategy and arbitrage opportunity in the market. Necessary and sufficient conditions are given for two essential properties of the market: arbitrage (Section 4) and completeness (Section 5). Several types of option contracts will be described (Section 6) such as European, American and some “exotic”, furthermore put and call options. Explanations of economic notions and examples are also included.

We shall see that the problem of option pricing can be considered as a special case of the valuation of contingent claims. Therefore, first we derive pricing formulas for contingent claims, generally, and secondly we obtain the results for option contracts which are known in the literature.

Our methods are based on martingale theory and the ideas of Harrison and Pliska [Harrison and Pliska 81]. Due to the probabilistic terms in which the results are formulated, one can easily understand some dominant features of the continuous markets by considering the analogy with the discrete time case. Though we mention that for some basic results probability theory is essentially not needed in the discrete time setting. (See [Dzhaparidze and Zuijlen 96].)

Finally, we will give a short summary of the general setting of the continuous time markets (Section 8).
2 Security Markets

2.1 General Assumptions

First of all, we shall describe the types of markets we are going to deal with. We consider financial markets in which two financial securities are traded and available any time for all actors (or participants) in the market (e.g., traders, speculators, investors). One of these securities is labelled "stock" and the other one is called "bond". We will assume that there are no transaction costs to be paid by the market participants for trading stocks or bonds.

The markets are studied over a finite time interval \([0, T]\) (\(T \in \mathbb{R}^+\)) where 0 represents the current date and \(T\) is interpreted as the terminal date. Here we distinguish between discrete time and continuous time models. We shall deal with the discrete time case where a finite sequence of moments is considered that we denote by \(0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T\). These moments are the trading times at which the new prices are announced in the market.

The bond is a riskless asset since its price \(B\) is a positive deterministic function of the time and the interest rate, where the interest rate is assumed to be known and non-random (but not necessarily constant) over \([0, T]\). In terms of economics, the bonds are debt claims, usually emitted by the state, banks or some other financial organizations in order to accumulate their capital. Bank accounts, government bonds, treasury bonds, corporate bonds, (zero) coupon bonds are examples of the notion at issue.

Unlike the bond, the stock is a risky asset whose price \(S\) is considered as a positive stochastic process over \([0, T]\). The stocks are usually shares issued by certain companies (e.g., Ltd.'s., joint-stock companies, etc.) to accumulate funds for their subsequent activity. It should be emphasized that the forthcoming mathematical models require only the fact that one of the two traded assets (called stock in our case) changes randomly while the other is non-random. Therefore not only ordinary shares but also foreign currency and other financial assets can satisfy the definition of the stock in our theory. In the case of ordinary shares, the influence of current state of the market and the correspondent production activity of the company are the main sources of the price randomness of the stock.

In the following, we shall denote the price of the bond and the price of the stock at time \(t_n\) \((n = 0, \ldots, N)\) by \(B_n\) and \(S_n\) respectively.

We will assume that borrowing and lending at the risk-free interest rate is possible. We mention that the bond price process is usually determined by the interest rate or vice versa. Namely, in most cases the bond price at time \(t_n\) \((n = 1, \ldots, N)\) is defined by \(B_n = (1 + r_n)B_{n-1}\), where \(r_n\) is the interest rate corresponding to the period \((t_{n-1}, t_n]\). Moreover, in case of a given bond price process one can define the bond's interest rate by the formulas \(r_n := \frac{B_n}{B_{n-1}} - 1\) \((n = 1, \ldots, N)\). In other words, this assumption means that borrowing and lending bonds is possible. We should like to emphasize that the bond price takes its new value at the trading times which means that the bond price is constant during the period \([t_n, t_{n+1})\) and equal to \(B_n\) \((n = 0, \ldots, N-1)\); thus \(B_{n+1}\) \((n = 1, \ldots, N)\) is valid from time point \(t_{n+1}\).

Similar assumptions are made for the stock. We suppose that the so-called short sale of stock is possible. It means that the participant sells the stock at some time point in \([0, T]\) but he or she does not have to deliver it until the terminal date \(T\). To give another interpretation one can imagine that the participant gets the amount \(S\) of money at time \(t \in [0, T]\), e.g. from a bank, to buy a share of stock but the participant's debt is expressed in terms of stock shares, i.e. at time \(T\) he will have to give back a share of stock which is equivalent to the amount \(S\) of money. Therefore the short sale can be seen as a "stock loan".

2.2 Trading in the market

One can become an actor in the market by selling or buying stock and bonds. In case \(\beta (\beta \in \mathbb{R})\) units of bonds and \(\gamma (\gamma \in \mathbb{R})\) units of stocks are held by the investor, we say that the investor possesses portfolio \(\pi\), which is defined by the couple \(\pi = (\beta, \gamma)\).

Let us imagine an investor having an initial capital \(X_0 \geq 0\), who is interested in increasing it in the future. For this purpose he invests the capital in stocks and bonds in such a way that he owns the portfolio \(\pi_1 = (\beta_0, \gamma_0)\) with \(X_0 = \beta_0 B_0 + \gamma_0 S_0\) at time \(t_0 = 0\), where \(S_0\) and \(B_0\) are the initial prices of the stock and bond respectively. Then, by selling bonds and buying stocks or vice versa during the time interval \((t_0, t_1]\) the investor might redistribute the portfolio \(\pi_0\) and transform it into a new portfolio \(\pi_1 = (\beta_1, \gamma_1)\). We shall assume that

- the transformation is made only on the basis of information available before time \(t_1\) in the market, that is only by considering the prices \(B_0\) and \(S_1\), and secondly, that
- neither inflow of capital (like dividends of stocks) nor outflow (like tax, consumption or transaction costs) takes place.

Therefore the redistributed portfolio must satisfy the equation

\[ X_0 = \beta_1 B_0 + \gamma_1 S_0 \]

before time \(t_1\). The assumption above is called the self-financing condition on the investment strategy.

At time \(t_1\) the new prices are announced in the market, so the investor possessing portfolio \(\pi_1\) realizes a new amount of capital, say \(X_1\), where

\[ X_1 = \beta_1 B_1 + \gamma_1 S_1.\]

Hence, due to the random change of the stock price, capital loss \((X_0 > X_1)\) might occur, but on the other hand profit can be obtained as well \((X_0 < X_1)\).

\[ \text{For conciseness, we will write 'he' instead of 'he or she' in this paper.} \]
In a similar way, given the known prices \( B_{n-1} \) and \( S_{n-1} \) at any arbitrary trading time \( t_{n-1} \) \((1 \leq n \leq N)\), the investor who owns portfolio \( (\beta_n, \gamma_n) \) possesses the capital \( X_{n-1} = \beta_n B_{n-1} + \gamma_n S_{n-1} \). Before time point \( t_n \) the investor can choose a new portfolio \( (\beta_n, \gamma_n) \) by considering the previous information (available at \( t_{n-1} \)) in such a way that the self-financing condition \( X_{n-1} = \beta_n B_{n-1} + \gamma_n S_{n-1} \) remains satisfied. This leads the investor to possess capital \( X_n = \beta_n B_n + \gamma_n S_n \) at time \( t_n \) when the new announcement of the prices is made in the market. In case of such a strategy, we say, that the sequence \( \{X_n\}_{n=1}^N \) is the corresponding value process of the strategy.

It should be emphasized that stock and bond prices are not allowed to take negative values whereas the entries of the portfolio owned by an investor may even be negative. A negative value of \( \beta \) corresponds to borrowing money at the bond’s interest rate, that is, getting a loan of amount \( x \) at time \( t \) \((t \in [t_{n-1}, t_n], 0 \leq n \leq N-1)\) means that the borrower will have to discharge the debt by paying \( x B_n / B_n \) at time \( T \). A negative value for \( \gamma \) represents a short sale of stocks. Finally we mention that one may assume not to allow the value process \( \{X_n\}_{n=1}^N \) described above to take values below zero during the period \([0, T]\) which seems fairly reasonable in realistic economic situations. However, as we shall see in the Sections Arbitrage and Market Completeness, this assumption is not needed in the forthcoming mathematical model.

In the mathematical model a monotone increasing sequence of \( \sigma \)-algebras, more precisely, a filtration is to represent the flow of information (see Definition 1): the further we go in time (or equivalently, the larger \( n \) is in \( t_n \)) the more pieces of information we get, and therefore the “richer” the \( \sigma \)-algebra of the possible events (measurable sets) will be.

### 2.3 Example

A simple example of discrete time markets is the so-called binary market. In a binary market the bond prices \( B_0, B_1, \ldots, B_N \) are given by the rule

\[
B_n = (1 + r_1)(1 + r_2) \cdots (1 + r_n) B_0 \quad (n = 1, \ldots, N)
\]
or equivalently

\[
B_n = (1 + r_n) B_{n-1} \quad (n = 1, \ldots, N)
\]

where \( N \) is the number of trading times and \( r_n \) is the interest rate in the market during the period \([t_{n-1}, t_n] \) for \( n = 1, \ldots, N \).

The stock price process starts from \( S_1 \) at time \( 0 \) and at each trading time it takes one of two possible values randomly. Usually the two values correspond to an upward jump and a downward jump of the price respectively. The evolution of the stock price can be interpreted by an oriented binary tree (see Figure 1) which has \( S_0 \) as its root and each vertex of layer \( n \) \((n = 0, \ldots, N)\) is a possible state of the price at time \( t_n \) and therefore each vertex.
Now the $n$'th layer of the tree (see Figure 1) is $(S_0(1 + \bar{a})^k (1 + \bar{b})^k | k = 0, \ldots, n)$. ($-1 < a < b$)

### 2.4 Further properties of the market

Another important economic term which should be mentioned here is the notion of arbitrage. In general, an arbitrage opportunity is some kind of riskless way of making profit and the person who executes a strategy in order to realize the profit by using the arbitrage opportunity is called the arbitrageur. A classical example is the case where an arbitrageur, noticing the different prices of the same product or currency or share of a company in two markets, enters simultaneously into transactions in the markets, for instance, by buying a number of those goods for a certain price in one of the markets and by selling them (almost) immediately at a higher price in the other market. One can also take advantage of the so-called interest arbitrage opportunity in which case the different interest rates in countries are the source of riskless profit.

Although arbitrage opportunities occur in every-day life, from a theoretical point of view our interest is focused on markets which exclude arbitrage opportunities. Therefore, further conditions will be needed on discrete time markets defined in Section Discrete Time Markets. Furthermore, by investigating this problem, one can also find necessary and sufficient conditions for market completeness which is an "ideal" property of the market. It means that for any desired wealth there exists a self-financing strategy such that it attains exactly this desired wealth at time $T$ if a certain amount of initial endowment is provided. The desired wealth does not need to be a constant value but it can be given in terms of the history of the stock price, i.e., it can be a function of $S_0, S_1, \ldots, S_N$. It is clear that any constant value (wealth) can be obtained by depositing its discounted value at the current time at the risk-free interest rate (i.e., by buying bonds).
3 Discrete Time Markets

3.1 The definitions of markets

Definition 1 (Discrete Time (B, S)_N Market) The set \( (\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{F}_n), B = (B_n), S = (S_n), N \) is called discrete time \((B, S)_N\) market \((N \in \mathbb{N})\) where

- \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space with filtration \( \mathcal{F} = (\mathcal{F}_n)_{n=0}^N \) where \([0, \Omega] = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{F}\),
- \(B = (B_n)_{n=0}^N\) is the price process of a bond with \(B_0 \in \mathbb{R}\), \(B_n > 0, n = 0, 1, \ldots, N\),
- \(S = (S_n)_{n=0}^N\) is the price process of a stock such that the \(S_n\)'s are positive \(\mathcal{F}_n\)-measurable random variables for \(n = 0, 1, \ldots, N\).

Remark 2 We stress that the price process of the bond is deterministic since the bond is a riskless asset.

Notation For convenience, we use the notation \(d.t.(B, S)_N\) for a discrete time market \((\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{F}_n), B = (B_n), S = (S_n), N\) and hence we will suppress the underlying probability space and filtration if it does not cause confusion.

Remark 4 In the definition above \(N\) is the number of trading times. Usually, as we shall assume, we have a \([0, T]\) time interval with \(0 = t_0 < t_1 < \cdots < t_N = T\) over which the market is examined and to which the option contracts are referred and therefore \(n \in \{1, \ldots, N\}\) corresponds to the \(n\)-th trading time \(t_n\). The time point \(T\) will be called the terminal date or expiration time (especially in case of investigating options).

Remark 5 For what follows, dealing with the discrete time case it is sufficient to assume that only a finite number of events may happen in the market (i.e., \(|\Omega| < \infty\)) and \(\mathbb{P}(\omega) > 0\) for all \(\omega \in \Omega\). More general cases are studied in the continuous time markets.

Definition 6 (Discrete Time (Nonhomogeneous) (B, S)_N Binary Market) A discrete time market \((\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{F}_n), B = (B_n), S = (S_n), N\) is called discrete time (nonhomogeneous) \((B, S)_N\) binary market with interest rates \(r_n\) and coefficients \(\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N\) if

- \(r_n \geq 0, -1 < a_n < b_n\) for \(n = 1, \ldots, N\),
- the bond price process \(B = (B_n)_{n=0}^N\) satisfies the equality
  \[ B_n = (1 + r_n)B_{n-1}, \quad \text{for} \quad n = 1, \ldots, N, \]
  \[ B_0 = 1, \]
  \[ S_n = (S_n)^N_{n=0}, \quad \text{satisfies the equality} \]
  \[ S_n = (1 + \rho_n)S_{n-1}, \quad \text{for} \quad n = 1, \ldots, N, \]
  where the \(\rho_n\)'s \((n = 1, \ldots, N)\) are independent random variables such that \(\{\rho_n \in \{a_n, b_n\}\} = \Omega\) and \(\rho_n \sim \mathbb{P}((\rho_n - b_n)/(a_n - b_n)) \in (0, 1)\) for \(n = 1, \ldots, N\) and finally,
- the filtration \(\mathcal{F} = (\mathcal{F}_n)\) is generated by \(\rho_1, \ldots, \rho_N\), that is \(\mathcal{F}_n = (0, \Omega)\) and \(\mathcal{F}_n = (\sigma(\rho_1, \ldots, \rho_n)) \) for \(n = 1, \ldots, N\).

Definition 7 (Discrete Time Homogeneous (B, S)_N Binary Market) A discrete time binary market with interest rates \(r_n\) and coefficients \(\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N\) is called discrete time homogenous \((B, S)_N\) binary market with interest rate \(r\) and coefficients \(a, b\) if the conditions

- \(r = r_n, a = a_n, b = b_n, n = 1, \ldots, N\),
- are satisfied and the random variables \(\rho_1, \ldots, \rho_N\) are \(i.d.\) (i.e. \(p = p_n\) for \(n = 1, \ldots, N\)).

Notation For convenience, we use the notation \(d.t.h.(B, S)_N\) for a discrete time (nonhomogeneous) binary market \((\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{F}_n), B = (B_n), S = (S_n), N\) with interest rates \(r_n\) and coefficients \(\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N\), whereas the notation \(d.t.h.(B, S)_N\) is to indicate that the market, at issue, is homogeneous. We will omit the underlying probability space, filtration, interest rates and coefficients if it does not cause misunderstanding.

Keeping in mind Remark 5, one can imagine \(\Omega\) as a bijective image of the space of realizations of the sequences \((\rho_1, \ldots, \rho_N)\) in case of a given binary market. It means that \(\Omega\) can be identified by the set

\[ \{ (x_1, \ldots, x_N) \mid x_i \in \{a_i, b_i\}, i = 1, \ldots, N \} \]

such that each \(\omega \in \Omega\) corresponds to a particular trajectory of the stock price binary tree (see Figures 1 and 2). In this interpretation it is obvious that the corresponding \(n\)-th \(\sigma\)-algebra \(\mathcal{F}_n\) \((n = 1, \ldots, N)\) can be identified with the set

\[ \{ (x_1, \ldots, x_N, y_1, \ldots, y_{N-n}) \mid y_i \in \{a_i, b_i\}, i = 1, \ldots, N-n \} \mid x_i \in \{a_i, b_i\}, i = 1, \ldots, n \}. \]

3.2 Strategies, hedging

Definition 9 In a \(d.t.(B, S)_N\) market let \(\beta_n\) and \(\gamma_n\) be \(\mathcal{F}_n\)-measurable random variables \((n = 1, \ldots, N)\) and \(\beta_1, \gamma_1 \in \mathbb{R}\). Then the sequence \(\pi := (\beta_n, \gamma_n)_{n=0}^N\) is called a strategy.

The sequence \(X^\pi_n := \beta_nB_n + \gamma_nS_n\) is the value process of \(\pi\) and \(\pi_n := (\beta_n, \gamma_n)\) is said to be the portfolio at trading time \(t_n\).
Remark 10 If an investor follows a strategy $\pi$ then the numbers $\beta_n$ and $\gamma_n$ show the number of bonds and stocks respectively held by the investor at time $t_n$. Although, it should be emphasized that the notion strategy does not have any realistic (every-day) meaning in this general way we defined it. Due to the "poor condition" in Definition 9, it cannot be considered as a well-known term of economics since some further restrictions and conditions (e.g. self-financing) are still needed to get a notion which can be interpreted as a strategy to be executed by an investor indeed.

Notation 11 Given a sequence $a_n \in \mathbb{R}$ for $n \in \mathbb{N}$ we write
\[ \Delta a_n := a_n - a_{n-1}, \quad n \in \mathbb{N}, \quad n \geq 1 \]
and we say that $\{\Delta a_n\}_{n=1}^{\infty}$ is the difference sequence of $\{a_n\}_{n=1}^{\infty}$.

Definition 12 In a d.t.-$(B, S)_N$ market a strategy $\pi = \{\pi_n = (\beta_n, \gamma_n)\}_{n=1}^{\infty}$ is self-financing if the equations
\[ X^\pi_{n+1} = \beta_n B_{n+1} + \gamma_n S_{n+1}, \quad n = 1, \ldots, N, \]
are satisfied.

Remark 13 In a d.t.-$(B, S)_N$ market
- in general we can write
\[ \Delta X^\pi_n = X^\pi_{n+1} - X^\pi_n = \beta_n B_{n+1} + \gamma_n S_{n+1} - (\beta_n B_{n+1} + \gamma_n S_{n+1}) = (\beta_n \Delta B_n + \gamma_n \Delta S_n) + (B_{n+1} \Delta \beta_n + S_{n+1} \Delta \gamma_n) \quad (n = 1, \ldots, N), \]
- in case of a self-financing strategy we have
\[ \Delta X^\pi_n = \beta_n \Delta B_n + \gamma_n \Delta S_n - (\beta_n \Delta B_n + \gamma_n \Delta S_n) \quad (n = 1, \ldots, N). \]

Summarizing our observations we obtain the following practical lemma.

Lemma 14 In a d.t.-$(B, S)_N$ market the following assertions for a strategy $\pi$ are equivalent:
(a) $\pi$ is a self-financing strategy, i.e. $X^\pi_{n+1} = \beta_n B_{n+1} + \gamma_n S_{n+1}$, $n = 1, \ldots, N$.
(b) $\Delta X^\pi_n = \beta_n \Delta B_n + \gamma_n \Delta S_n$, $n = 1, \ldots, N$.
(c) $B_{n+1} \Delta \beta_n + S_{n+1} \Delta \gamma_n = 0$, $n = 1, \ldots, N$.

Lemma 15 explains us the self-financing condition from different points of view. After the new prices are announced in the market at time $t_n$, one can make changes in his portfolio (that is determine $\pi_n$) provided that $X^\pi_{n+1}$ is the given capital for the trading plans (see (a)). These changes might be said to be internal or interior in order to emphasize that neither outflows (like tax payments, operation costs, other consumptions, etc.) nor inflows (like stock dividends, other income or capital to be reinvested in the strategy, etc.) are supposed in our model as it is stressed in (a) and (c). Therefore, profit can be gained only due to the changes of prices in the market (see (b)).

Remark 16 Referring to the random variables $\rho_n(\omega), S_n(\omega), \beta_n(\omega), \gamma_n(\omega), X_n(\omega)$ and the random vector $\pi_n(\omega) = (\beta_n(\omega), \gamma_n(\omega))$, for convenience, we shall omit their dependence on $\omega \in \Omega$ and we shall simply write $\rho_n, S_n, \beta_n, \gamma_n, X_n = (\beta_n, \gamma_n)$ instead.

Definition 16 In a d.t.-$(B, S)_N$ market let $z \in \mathbb{R}$ and suppose that $f_N$ is a Borel function such that $f_N : \mathbb{R}^{N+1} \mapsto \mathbb{R}$. A strategy $\pi = \{\pi_n\}_{n=1}^{\infty}$ is called an $(x, f_N)$-hedge (or hedging strategy) if
\[ X^\pi_N(\omega) = x, \quad \forall \omega \in \Omega, \quad (1) \]
and
\[ X^\pi_N(\omega) \geq f_N(S_1(\omega), \ldots, S_N(\omega)), \quad \forall \omega \in \Omega. \quad (2) \]
If $X^\pi_N(\omega) = f_N(S_1(\omega), \ldots, S_N(\omega)), \forall \omega \in \Omega$, then we say that $\pi$ is a minimal $(x, f_N)$-hedge. We denote by $\Pi(x, f_N)$ the set of all self-financing $(x, f_N)$-hedges.

Remark 17 Now suppose that for the function $f_N$ in Definition 16 there exists a Borel function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $f_N(S_1(\omega), \ldots, S_N(\omega)) = g(S_N(\omega))$ for all $\omega \in \Omega$. Having an $(x, f_N)$-hedge executed by the investor in order to obtain at least the capital $g(S_N)$ we say that the investor organizes a strategy against (the wealth or capital) $g(S_N)$.

Definition 18 In a d.t.-$(B, S)_N$ market with a Borel function $f_N : \mathbb{R}^{N+1} \mapsto \mathbb{R}$
\[ C_{N, f_N} := \inf\{x > 0 \mid x \in \Pi(x, f_N) \neq \emptyset\} \]
is said to be the investment cost, guaranteeing at time $t_N$, a capital not less than $f_N(S_1(\omega), \ldots, S_N(\omega))$, $\forall \omega \in \Omega$.

Lemma 19 For any d.t.-$(B, S)_N$ market and a Borel function $f_N : \mathbb{R}^{N+1} \mapsto \mathbb{R}$ there exists $z \in \mathbb{R}^+$ such that $\Pi(x, f_N) \neq \emptyset$.
For instance, consider
\[ x := \frac{B_0}{B_N} \max_{\omega \in \Omega} |f_N(S_0, S_1(\omega), \ldots, S_N(\omega))| \]
with strategy $\pi = \{\pi_n\}_{n=1}^{\infty}$ where
\[ \pi_n := (\beta_n, \gamma_n) \equiv (x/B_0, 0), \quad \text{for } n = 0, \ldots, N. \]
Then $\pi \in \Pi(x, f_N)$ and therefore we have $C_{N, f_N} < \infty$. 

\[ 13 \]

\[ 14 \]
Remark 20 It is obvious that $C_{N,f_{X}}$ is defined in order to indicate the minimal amount of initial capital which guarantees that the investor realizes capital $X^*_{N}$ at time $T$ with the property $X^*_{N}(\omega) = f_{N}(S_{0}(\omega), \ldots, S_{N}(\omega))$ with respect to a certain strategy $\pi$. Note that we have not shown (yet) the existence of such kind of strategy (see Remark 39 and Theorem 40).

Remark 21 In economics it is common and reasonable to add the condition $X^*_{N} \geq 0$ ($0 \leq n \leq N$) to the definition of hedging strategies. We shall show later that having a strategy for a certain initial capital $x$ and "target" function $f_{N}$ implies the existence of a nonnegative strategy for the same initial capital $x$ and "target" function (see Remark 39).

Definition 22 In a ${d,t}$-$(B,S)_{N}$ market, given a strategy $\pi$ the process

$$M^*_n := \frac{X^*_n}{B_n} \quad (0 \leq n \leq N)$$

(3)

is said to be the discounted value process of the strategy.

It is clear that (3) is the natural way of discounting (according to the known interest rate) in economics in order to compare the capitals corresponding to different time points. (In our case, furthermore, it is normalized by the initial bond price $B_0$.)

4 Arbitrage

4.1 The notion of arbitrage

To put it in a fairly general way, in economics, arbitrage opportunity may be any riskless way of making profit. In this context it is natural to define an arbitrage strategy as follows.

Definition 23 In a $d,t$-$(B,S)_{N}$ market a self-financing strategy $\pi$ is said to be arbitrage, or an arbitrage strategy if

- $X^*_0 = 0$.
- $X^*_n \geq 0$ for $1 \leq n \leq N$.
- $\exists \omega \in \Omega : X^*_N(\omega) > 0$ (that is $\mathbb{P}(X^*_N > 0) > 0$).

We say that the market excludes arbitrage (opportunities), if there is no self-financing arbitrage strategy on the market.

4.2 Martingale measure

Definition 24 In a discrete time market, say $\{\Omega, \mathcal{F}, \mathbb{P}=(\mathcal{F}_n), B=(B_n), S=(S_n), N\}$. $\mathbb{P}^*$ is said to be an equivalent martingale measure if

- $\mathbb{P}^*$ is a probability measure on $\Omega, \mathcal{F}$.
- $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent and
- the sequence $(S_n/B_n, \mathcal{F}_n, \mathbb{P}^*)_{0 \leq n \leq N}$ forms a martingale.

Lemma 25 In a $d,t$-$(B,S)_{N}$ market $(S_n/B_n, \mathcal{F}_n, \mathbb{P}^*)_{0 \leq n \leq N}$ is a martingale iff $(X^*_n/B_n, \mathcal{F}_n, \mathbb{P}^*)_{0 \leq n \leq N}$ is a martingale for any self-financing strategy $\pi$.

Proof. $X^*_n/B_n = \beta_n + \gamma_n S_n/B_n$ and we have

$$\mathbb{E}^* (X^*_n/B_n | \mathcal{F}_{n-1}) = \beta_n + \gamma_n \mathbb{E}^* (S_n/B_n | \mathcal{F}_{n-1}) = X^*_{n-1}/B_{n-1},$$

since $\beta_n$ and $\gamma_n$ are $\mathcal{F}_{n-1}$-measurable (where $\mathbb{E}^*$ denotes the expected value with respect to $\mathbb{P}^*$). \hfill \Box

Notation 26 Given a probability measure $\mathbb{P}^*$, we will denote the expected value with respect to $\mathbb{P}^*$ by $\mathbb{E}^*$ in this paper.
Theorem 27 Suppose that the inequalities $a_n < r_n < b_n$ hold for $n = 1, \ldots, N$ in a d.t.d.-
$(B, S)_N$ market with interest rates $\{r_n\}_{n=1}^N$ and coefficients $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$. Then the measure $\mathbb{P}'$ which satisfies
\[ \mathbb{P}'(\omega) = \frac{r_n - a_n}{b_n - a_n} \prod_{n=1}^{N-1} \frac{b_n - r_n}{b_n - a_n} \]
for $n = 1, \ldots, N$, is an equivalent martingale measure in the market.

Remark 28 It is clear that equation (4) defines the measure $\mathbb{P}'$ uniquely. Indeed, the σ-
algebra $\mathcal{F} = \mathcal{F}_0$ is generated by the independent random variables $\rho_1, \ldots, \rho_N$. Recalling the fact that $\Omega$ can be imagined as a set of realizations of $\rho_1, \ldots, \rho_N$ (see page 12) it is easy to see that for an element $\omega \in \Omega$.
\[ \mathbb{P}'(\omega) = \prod_{\rho_n = b_n} \left( \frac{r_n - a_n}{b_n - a_n} \right) \prod_{\rho_n = a_n} \left( \frac{b_n - r_n}{b_n - a_n} \right) \]

must be valid where the vector $(x_1, \ldots, x_N)$ is the corresponding realization of the event $\omega$. Particularly,
\[ \mathbb{P}'(\omega) = \left( \frac{r_n - a_n}{b_n - a_n} \right)^k \left( \frac{b_n - r_n}{b_n - a_n} \right)^{N-k} \]
in a homogeneous binary market where $k$ is the number of upward jumps of the stock price during $[0, T]$ if the price evolves along the trajectory corresponding to event $\omega$. Thus $\mathbb{P}'$ in (5) is a binomial probability measure and that is why the homogeneous binary markets are sometimes called binomial (security) markets in the literature.

Proof of Theorem 27. Notice that
\[ \mathbb{E}'(\rho_n - a_n(1 - \rho_n) + b_n\rho_n) = \frac{(b_n - a_n)\rho_n + a_n - r_n}{b_n - a_n} \quad \text{for} \quad n = 1, \ldots, N. \]

Thus by the fact that the stock price process is adapted to the filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n=0}^N$ we get
\[ \mathbb{E}'\left( \frac{S_n}{B_n} \right| \mathcal{F}_{n-1} \bigg) - \frac{S_{n-1}}{B_{n-1}} \mathbb{E}'(\rho_n + 1) | \mathcal{F}_{n-1} \bigg) \]
\[ = \frac{S_{n-1}}{B_{n-1}} \left( \frac{r_n + 1}{b_n - a_n} \right), \quad n = 1, \ldots, N, \]
which completes the proof.

Notation 29 In this paper $I_A(\cdot)$ is used for the indicator function of the set $A$.

Lemma 30 Let $\xi_n (n = 0, \ldots, N, N \in \mathbb{N})$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F} = (\mathcal{F}_n)_{n=0}^N$.
If
- $\mathbb{E}[\xi_n] < \infty \quad (n = 0, \ldots, N)$,
- $\xi_n$ are $\mathcal{F}_n$-measurable \quad $(n = 0, \ldots, N)$, and
- for any Markov-time $\tau : \Omega \to \{0, 1, \ldots, N\}$ the property $\mathbb{E}\xi_\tau = \mathbb{E}\xi_0$ holds,
then $(\xi_0, \mathcal{F}_n, \mathbb{P})_{0 \leq n \leq N}$ forms a martingale.

Proof. Let $A \in \mathcal{F}_n$ and let the Markov-time $n_A$ be defined as follows
\[ n_A(\omega) := \begin{cases} n & \text{if } \omega \in A \\ N & \text{otherwise} \end{cases} \]

Then we get
\[ \mathbb{E}\xi_n = \mathbb{E}\xi_{n_A} = \mathbb{E}\xi_0 I_A + \mathbb{E}\xi_N I_\tau, \]
in particular,
\[ \mathbb{E}\xi_0 = \mathbb{E}\xi_N. \]

Therefore, from the last two equations we get
\[ \int \xi_n d\mathbb{P} = \mathbb{E}\xi_N I_A = \mathbb{E}\xi_N - \mathbb{E}\xi_N I_\tau = \mathbb{E}\xi_0 I_A = \int \xi_0 d\mathbb{P}, \]
that is $\xi_n = \mathbb{E}(\xi_n | \mathcal{F}_n)$ a.s. which implies with the aid of the tower law that
\[ \mathbb{E}(\xi_{n+1} | \mathcal{F}_n) = \mathbb{E}(\mathbb{E}(\xi_{n+1} | \mathcal{F}_n)) = \mathbb{E}\xi_{n+1} - \mathbb{E}\xi_n \quad \text{a.s.} \]

4.3 The main assertions on arbitrage

Theorem 31 In a d.t.d.-$(B, S)_N$ market the following statements are equivalent:
- (1) there exists an equivalent martingale measure,
- (2) the market excludes arbitrage opportunities.
Proof. (1) ⇒ (2) Suppose that \( \mathbb{P}^* \) is an equivalent martingale measure. Let us assume that \( \pi \) is a self-financing strategy with initial capital \( X_{\pi}^0 = 0 \). Then by Lemma 25
\[
\mathbb{E} X_{\pi}^N = \frac{B_N}{B_0} X_{\pi}^0 - 0
\]
and therefore \( \pi \) cannot arbitrage strategy since \( X_{\pi}^N \geq 0 \) and \( \mathbb{P}^*(X_{\pi}^N > 0) > 0 \) would imply \( \mathbb{E} X_{\pi}^N > 0 \).

(2) ⇒ (1) Suppose that there is no arbitrage strategy in the market and let
\[
V_i := \{ \xi : \Omega \mapsto \mathbb{R} \text{ r.v. } | \exists \text{ self-financing strategy } \pi : X_{\pi}^0 = 0 \text{ and } X_{\pi}^N = \xi \}
\]
and
\[
V_1 := \{ \xi : \Omega \mapsto \mathbb{R} \text{ r.v. } | \xi \geq 0 \text{ and } \mathbb{E} \xi \geq 1 \}.
\]
For easy understanding, we get the underlying statement in 5 steps.

Step 1. We shall show that \( V_0 \cap V_1 = \emptyset \).
Suppose that \( \exists \xi \in V_0 \cap V_1 \). Then there exists a self-financing strategy \( \pi \) such that \( X_{\pi}^0 = 0 \) and \( X_{\pi}^N = \xi \) which implies – as we shall show – the existence of an arbitrage strategy \( \exists \) defined as follows which contradicts assumption (2).

If \( X_{\pi}^n \geq 0 \) (0 ≤ n ≤ N) then \( \exists \pi \) is an arbitrage strategy.
Otherwise, \( \exists N < N \) and \( \omega_0 \in \Omega \) such that \( X_{\pi}^N(\omega_0) < 0 \) and \( X_{\pi}^N(\omega) \geq 0 \) \( \forall \omega \in \Omega \) and \( n > m \). Then an arbitrage strategy \( \{ \pi_m = (\beta_m, \gamma_m) \}_{m=0}^n \) can be constructed as follows:
\[
\beta_m(\omega) := I_{(\omega=\omega_0)} I_{(m>m)} \left( \beta_m(\omega) - \frac{X_{\pi}^m(\omega)}{B_m} \right),
\]
\[
\gamma_m(\omega) := I_{(\omega=\omega_0)} I_{(m>m)} \gamma_m(\omega) \quad \text{for } m = 0, \ldots, N.
\]

First, we check the self-financing property of \( \exists \). If \( n \leq m \) or \( \omega \neq \omega_0 \) then \( \Delta \beta_m(\omega) = \Delta \gamma_m(\omega) = 0 \).

If \( n = m + 1 \) then
\[
\Delta \beta_m(\omega) - \beta_{m+1}(\omega_0) - \frac{X_{\pi}^m(\omega_0)}{B_m}
\]
and
\[
\Delta \gamma_m(\omega) = \gamma_{m+1}(\omega_0).
\]
so that
\[
B_{m+1} \Delta \beta_m(\omega_0) + S_{m+1} \Delta \gamma_m(\omega_0) = \left( \beta_{m+1}(\omega_0) - \frac{X_{\pi}^m(\omega_0)}{B_m} \right) B_m + \gamma_{m+1}(\omega_0) S_m
\]
and
\[
X_{\pi}^m(\omega_0) - X_{\pi}^m(\omega_0) = 0.
\]
If \( n > m + 1 \) then \( \Delta \beta_m(\omega_0) = \Delta \beta_m(\omega_0) = \Delta \gamma_m(\omega_0) \) which implies \( B_{m+1} \Delta \beta_m(\omega_0) + S_{m+1} \Delta \gamma_m(\omega_0) = 0 \) since \( \pi \) is self-financing and we conclude that \( \exists \) is self-financing.

Secondly, \( X_{\pi}^N \geq 0 \) (0 ≤ n ≤ N) since for \( n > m \) we have
\[
X_{\pi}^N(\omega) - \beta_m(\omega) B_m + \gamma_m(\omega) S_m = I_{(\omega=\omega_0)} \left( \beta_m(\omega) B_m + \gamma_m(\omega) S_m - \frac{X_{\pi}^m(\omega)}{B_m} B_m \right) \geq 0
\]
and it is clear that \( X_{\pi}^N(\omega) = 0 \) for \( n < m \). Moreover, \( \exists \omega \in \Omega \) s.t. \( X_{\pi}^N(\omega) > 0 \), namely \( \omega_0 \), since
\[
X_{\pi}^N(\omega_0) = \beta_N(\omega_0) B_N - \frac{X_{\pi}^N(\omega_0) B_N}{B_m} + \gamma_N(\omega_0) S_N
\]
and
\[
X_{\pi}^N(\omega_0) = \beta_N(\omega_0) B_N - \frac{X_{\pi}^N(\omega_0) B_N}{B_m} > 0.
\]

Step 2. Let \( f : V_0 \cup V_1 \mapsto \mathbb{R}^k \) denote the bijection \( f(\xi) = (\xi(\omega_1), \ldots, \xi(\omega_k)) \) for \( \xi \in V_0 \cup V_1 \) where \( k = |\mathbb{E} \xi| = |\omega_0, \ldots, \omega_k| \).

The set \( f(V_0) \) is a linear subspace of \( \mathbb{R}^k \) since \( \xi, \eta \in V_0 \) we have \( \lambda_1 \xi + \lambda_2 \eta \in V_0 \) \( \forall \lambda_1, \lambda_2 \in \mathbb{R} \) because the strategy \( \pi = \lambda_1 \pi_1 + \lambda_2 \pi_2 = (\lambda_1 \beta_1 + \lambda_2 \beta_2)I_{(\omega=\omega_0)} + \lambda_2 \beta_2 \) produces the capital \( X_{\pi}^N = \lambda_1 \beta_1 + \lambda_2 \beta_2 \eta \) at time \( t_N \) where \( \pi_1 \) and \( \pi_2 \) are strategies satisfying the definition of \( V_0 \) respectively.

The subset \( f(V_1) \) of \( \mathbb{R}^k \) is convex since given \( \xi, \eta \in V_1 \) and \( \lambda \in [0,1] \) it is clear that \( \lambda \xi + (1 - \lambda) \eta \geq 0 \) and \( \mathbb{E}(\lambda \xi + (1 - \lambda) \eta) \geq \lambda + (1 - \lambda) \).

Step 3. The construction of \( \mathbb{P}^* \).
It is known in linear algebra that given the linear subspace \( f(V_0) \) and the convex subset \( f(V_1) \) in \( \mathbb{R}^k \) with empty intersection there exists a real valued linear function \( l \) on \( \mathbb{R}^k \) such that
\[
l(v) = 0 \quad \text{if } v \in f(V_0)
\]
\[
l(v) > 0 \quad \text{if } v \in f(V_1).
\]
Furthermore, the linear function \( l \) corresponds to a vector \( q \in \mathbb{R}^k \) so that it can be written in the form \( l(v) = \langle v, q \rangle = \sum_{i=1}^k v_i q_i \) where \( v = (v_1, \ldots, v_k) \in \mathbb{R}^k \), \( q = (q_1, \ldots, q_k) \in \mathbb{R}^k \).
Let $\xi_1, \ldots, \xi_k$ are random variables defined by
\[ \xi_i(\omega_j) := \begin{cases} \frac{1}{P(\{\omega_j\})} & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} \]
where $i, j = 1, \ldots, k$. Then $\xi_i \geq 0$ and $EX_i = 0$, hence $\xi_i \in V_1$ (1 $\leq i \leq k$) and we get $I(f(\xi_i)) = q_j/P(\{\omega_j\}) > 0$ from which it follows that $q_j > 0$ (1 $\leq i \leq k$). Now, let us define $P'$ as follows
\[ P'(\{\omega_j\}) := \frac{q_j}{\sum_{j=1}^{k} q_j} \quad (i = 1, \ldots, k). \]

Step 4. Next we notice that a self-financing strategy $\pi$ with initial capital $X_0^\pi = 0$ has the property $E X_N^\pi = 0$. It can easily be seen, since $X_N^\pi \in V_0$ and hence $0 = \{f(X_N^\pi) = \sum_{n=1}^{N} X_N^\pi(\omega_i)q_i = E X_N^\pi \sum_{i=1}^{N} q_i. $

Step 5. We shall verify that $P'$ is an equivalent martingale measure indeed.
By the finiteness of $\Omega$ and $P(\{\omega_i\}) > 0$, $P'(\{\omega_i\}) > 0$ for $i = 1, \ldots, k$ it is trivial that $P'$ and $P$ are equivalent measures. By Lemma 30 it is sufficient to show that for an arbitrary Markov time $\tau : \Omega \rightarrow \{0, \ldots, N\}$ with respect to $P$ the property
\[ E' \left( \frac{S_n}{B_n} - \frac{S_0}{B_0} \right) = 0 \]
holds, therefore $(S_n/B_n, F_n, P')_{n=0,\ldots,N}$ is a martingale. Let $\tau$ be a Markov time as above and define
\[ \beta_n := \frac{S_n}{B_n} I_{[n+1]} - \frac{S_0}{B_0} \]
\[ \gamma_n := I_{[n+1]} \]
and
\[ \pi_n := (\beta_n, \gamma_n) \quad \text{for } n = 0, \ldots, N. \]
Then the strategy $\pi = (\pi_n)_{n=0}^N$ starts with capital zero at time $t_0$ since
\[ X_0^\pi = \frac{S_0}{B_0} B_0 = 0. \]
To check the self-financing property one can consider
\[ B_{n-1} I_{[0]}(\tau) + S_{n-1} I_{[n+1]}(\tau) = \frac{S_n}{B_n} (I_{[0]}(\tau) + I_{[n+1]}(\tau)) B_{n-1} + (I_{[1]}(\tau) - I_{[2]}(\tau) - I_{[2]}(\tau)) S_{n-1} \]
\[ = \frac{S_n}{B_n} I_{[n+1]}(\tau) B_{n-1} - I_{[n+1]}(\tau) S_{n-1} = 0. \]
By the application of step 4
\[ 0 = E X_N^\pi = E' (\beta_N B_N + \gamma_N S_N) \]
\[ = E' \left( \left( \frac{S_n}{B_n} I_{[n+1]}(\tau) - \frac{S_0}{B_0} \right) B_N + \frac{S_n}{B_n} I_{[n+1]}(\tau) B_{n-1} \right) \]
and thus equation (6) holds which completes the proof.

Theorem 31 plays an important role in option theory and generally in economics. It states a fortunate coincidence of mathematical and economic notions. From an economic point of view investigating arbitrage and markets excluding arbitrage is essential (e.g. in the theory of balanced economies) and on the other hand, we will see that the existence of an equivalent martingale measure gives us an excellent mathematical tool to handle calculations in an easy way.

It is also important that the real measure of the market $P$ is unknown which is quite usual in the theory of probability and statistics unlike the nature of our pricing problem which is fairly non traditional. Namely, we are interested in developing formulas for $C_{N,T}$ (see Definition 18) under certain circumstances and notice that the definition of $C_{N,T}$ is based on the notion of hedging strategy which is independent of any measure of the market, moreover the hedging conditions (1), (2) must be valid for all possible event of the market (see Definition 16). That is why some problems can be handled without the theory of probability (see [Dzhaparidze and Zuijlen 96]) as we have already mentioned.

Corollary 22 In a d-dimensional Brownian motion market with interest rates $\{r_n\}_{n=1}^N$ and coefficients $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$, the following statements are equivalent:

1. there exists an equivalent martingale measure.
2. the market excludes arbitrage opportunities.
3. $a_n < r_n < b_n$ holds for $n = 1, \ldots, N$.

Proof. Due to Theorem 27 and Theorem 31 the only statement needed to be proved is the implication (3) $\Rightarrow$ (2) what we show in an indirect way.
Suppose that there exists an integer $1 \leq n' \leq N$ such that $r_{n'} \notin (a_{n'}, b_{n'})$.
For instance, consider the case $r_{n'} \leq b_{n'}$. Then choose the strategy $\pi$ defined as follows.

$$
\beta_k := \begin{cases} 
0 & \text{if } 0 \leq k \leq n' - 1 \\
\frac{S_{n'|B_{n'}} - \bar{a}_{n'}}{B_{n'}} & \text{if } k = n' \\
\frac{S_{n'|B_{n'}} - \bar{a}_{n'}}{B_{n'}} & \text{otherwise}
\end{cases}
$$

$$
\gamma_k := \begin{cases} 
0 & \text{if } 0 \leq k \leq n' - 1 \\
1 & \text{if } k = n' \\
0 & \text{otherwise}
\end{cases}
$$

$$
\pi := ((\beta_a \gamma_a))_{k=0}^{n'}
$$

Notice that the strategy $\pi$ is self-financing. Moreover,

$$
X_k^* = -\frac{S_{n'|B_{n'}}}{B_{n'-1}} B_{n'} + S_{n'} = (-1 + r_{n'}) (1 + \rho_{n'}) S_{n'-1} \geq (-1 + r_{n'}) (1 + \rho_{n'}) S_{n'-1} > 0
$$

and

$$
X_n^* = \beta_n B_n = \frac{B_n}{B_{n'}} X_{n'} > 0,
$$

that is $\pi$ is an arbitrage strategy which contradicts (2).

A similar argument can be used for the case $r_{n'} \geq b_{n'}$ where $-\pi$ could be an example for arbitrage.

5 Market Completeness

As we have already mentioned, in a complete market any desired wealth, which might depend on the history of the stock prices, can be obtained by executing a certain self-financing strategy. We define this notion in our model as follows.

**Definition 33** A d.t.-$(B, S)_N$ market is called complete if for any random variable $\xi$ there exist a self-financing strategy $\pi$ such that

$$
X_k^*(\omega) = \xi(\omega) \quad \text{for } \forall \omega \in \Omega.
$$

**Theorem 34** Suppose that there exists an equivalent martingale measure $\mathbb{P}$ in a d.t.-$(B, S)_N$ market. Then the following statements are equivalent:

1. the market is complete,
2. $\mathbb{P}$ is the only equivalent martingale measure in the market,
3. each martingale $(M_n, \mathcal{F}_n, \mathbb{P})_{n\in\mathbb{N}}$ admits the representation

$$
M_n = M_0 + \sum_{k=1}^{n} \gamma_k \Delta m_k \quad \text{for } n = 1, \ldots, N
$$

where the $\gamma_k$'s are $\mathcal{F}_{n-1}$-measurable random variables $(n=1, \ldots, N)$ and

$$
m_n := \frac{S_n}{B_n} \quad \text{for } n = 1, \ldots, N.
$$

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\mathbb{P}^\ast$ is an equivalent martingale measure as well. We will show that $\mathbb{P}^\ast = \mathbb{P}$.

Let $A$ be an arbitrary $\mathcal{F}$-measurable set and choose $\xi(\omega) := I_A(\omega)$. Then by the completeness of the market there exists a self-financing strategy $\pi$ with $X_k^* = \xi$. The discounted value process of $\pi$ is a martingale with respect to any equivalent martingale measure. Particularly,

$$
\mathbb{E}^\pi \frac{X_n}{B_n} = \frac{X_0}{B_0} - \mathbb{E}^\ast \frac{X_n}{B_n}
$$

where $\mathbb{E}^\pi$ and $\mathbb{E}^\ast$ denote the expectation with respect to the measures $\mathbb{P}^\pi$ and $\mathbb{P}^\ast$ respectively, hence

$$
\mathbb{P}^\pi(A) = \mathbb{E}^\pi I_A = \mathbb{E}^\ast I_A = \mathbb{P}^\ast(A)
$$

which completes the proof of implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Notice that the uniqueness of the equivalent martingale measure $\mathbb{P}$ implies that $\mathbb{P}^\ast$ must be the one constructed in the proof of Theorem 31.
We shall use the same notation that we had in the proof of implication \((2) \Rightarrow (1)\) in Theorem 31.

Recall that
\[ V_1 := \{ \xi : \Omega \mapsto \mathbb{R} \text{ r.v. } \exists \text{ self-financing strategy } \pi : X_0^\pi = 0 \text{ and } X_N^\pi = -\xi \} \]
and define
\[ V_2 := \{ \xi : \Omega \mapsto \mathbb{R} \text{ r.v. } \mathbb{E}^\xi = 0 \}. \]
The set \( f(V_2) \) (see page 20 for the definition of \( f \)) is clearly a linear subspace of \( \mathbb{R}^k \) since \( \mathbb{E}^\xi \) is a linear functional over the set of the random variables on the underlying probability space of the market. Step 4 on page 21 says that \( V_0 \subseteq V_2 \). First we show in steps (a), (b) and (c) that these two subspaces coincide, i.e., \( V_0 = V_2 \).

Step (a). Let us assume that \( V_0 \neq V_2 \) and hence there exists a nonzero vector \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_k) \in f(V_2) \) which is orthogonal to the subspace \( f(V_1) \) of the linear space \( f(V_2) \) in \( \mathbb{R}^k \), i.e.,
\[ \langle \tilde{x}, x \rangle = 0 \quad \text{for all } x \in f(V_0). \]

Now, keeping in mind that \( q_i > 0 \) for \( i = 1, \ldots, k \) in the construction of \( \mathbb{P}^* \), we can choose a sufficiently small \( \varepsilon > 0 \) such that
\[ \tilde{q}_i := q_i - \varepsilon \tilde{x}_i > 0 \]
for \( i = 1, \ldots, k \). Let \( \tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_k) \) and notice that
\[ \langle \tilde{q}, x \rangle - \langle q, x \rangle = \varepsilon \langle \tilde{x}, x \rangle = 0 \quad \text{for } x \in f(V_1). \]

Step (b). Define
\[ \tilde{\mathbb{P}}(\omega) = \frac{\tilde{q} \mathbb{P}(\omega)}{\sum_{j=1}^k q_j} \quad \text{for } i = 1, \ldots, k. \]
The measure \( \tilde{\mathbb{P}} \) is an equivalent martingale measure in the market what we show in the same way as we did for \( \mathbb{P}^* \) in step 5 on page 21. For this let \( \tau \) be an arbitrary Markov time and recall the self-financing strategy \( \pi \) defined in step 5 on page 21. Let us denote the expected value with respect to \( \tilde{\mathbb{P}} \) by \( \tilde{\mathbb{E}} \). Then the functional \( \tilde{\mathbb{E}} \) vanishes over the set \( V_0 \) due to statement (7) and hence we have
\[ 0 = \tilde{\mathbb{E}} X_N^\pi = \tilde{\mathbb{E}}(\beta N + \gamma N S_N) \]
\[ = \tilde{\mathbb{E}} \left( \frac{S_i}{B_i} I_{i \neq \tilde{N} \in \mathbb{N}} + \frac{S_{\tilde{N}}}{B_{\tilde{N}}} B_{\tilde{N}} + \frac{S_i}{B_i} I_{i = \tilde{N}} B_N \right) \]
\[ = B_N \tilde{\mathbb{E}} \left( \frac{S_i}{B_i} - \frac{S_{\tilde{N}}}{B_{\tilde{N}}} \right) \]
which together with Lemma 30 implies that \( \tilde{\mathbb{P}} \) is an equivalent martingale measure indeed.

Step (c). Because of the uniqueness of \( \mathbb{P}^* \) we get \( \mathbb{P}^* = \tilde{\mathbb{P}} \) which is equivalent to
\[ q = a \tilde{q} = a(q - \varepsilon \tilde{x}). \]
where \( a = \sum_{j=1}^k q_j / \sum_{j=1}^k \tilde{q}_j \). Hence
\[ (1 - a) \tilde{q} = -a \varepsilon \tilde{x}. \]  
(8)

But \( \tilde{x} \in f(V_2) \) which together with the fact that \( \mathbb{E}^\xi \) vanishes over \( V_2 \) implies that the equality (8) is possible only if \( a = 1 \) and \( \tilde{x} \) is a zero vector. This is a contradiction and therefore \( V_1 = V_2 \) is proved.

Step (d). Finally we show that the completeness of the market follows from the coincidence of \( V_0 \) and \( V_2 \). Let \( \xi \) be an arbitrary \( (\mathcal{F}_n \text{-measurable}) \) random variable. Then the random variable \( \xi - \mathbb{E}^\xi \) is an element of \( V_2 \) and hence it is in \( V_1 \) as well. Thus there exists a self-financing strategy \( \pi = \{(\beta_n, \gamma_n)\}_{n=0}^N \) with
\[ X_0^\pi = 0 \quad \text{and} \quad X_N^\pi = \xi - \mathbb{E}^\xi. \]

Define
\[ \beta_n := \beta_n + \frac{\mathbb{E}^\xi}{B_N} \quad \text{for } n = 0, \ldots, N. \]
Then the strategy \( \pi' = \{(\beta'_n, \gamma'_n)\}_{n=0}^N \) is clearly self-financing with
\[ X_N^{\pi'} = \xi \]
which completes the proof of the implication.

(1) \( \Rightarrow \) (3) Suppose that the market is complete and let \( (M_n, \mathcal{F}_n, \mathbb{P})_{0 \leq n \leq N} \) be a martingale.

Then it follows from the completeness that there exists a self-financing strategy \( \pi = \{(\beta_n, \gamma_n)\}_{n=0}^N \) such that
\[ X_N^\pi(\omega) = B_N M_N(\omega). \]

Due to the fact that \( \mathbb{P}^* \) is an equivalent martingale measure we get that \( (M_n^*, \mathcal{F}_n, \mathbb{P}^*)_{0 \leq n \leq N} \) is a martingale and therefore
\[ M_n = \mathbb{E}(M_N | \mathcal{F}_n) = \mathbb{E}(X_N^\pi | \mathcal{F}_n) = \mathbb{E}(M_N^\pi | \mathcal{F}_n) = M_n^\pi. \]

Thus we get finally that
\[ M_n - M_{n+1} = M_n^\pi - M_{n+1}^\pi = \frac{\beta_n B_n + \gamma_n S_n}{B_n} - \frac{\beta_{n+1} B_{n+1} + \gamma_{n+1} S_{n+1}}{B_{n+1}} \]
\[ = \frac{\beta_n B_n + \gamma_n S_n}{B_n} - \frac{\beta_{n+1} B_{n+1} + \gamma_{n+1} S_{n+1}}{B_{n+1}}. \]
for $n = 1, \ldots, N$ which means that the form
\[ M_n = M_0 + \sum_{k=1}^{n} \gamma_k \Delta m_k, \quad n = 1, \ldots, N, \]
is valid indeed.

(3) $\Rightarrow$ (1) Let $\xi$ be an $(\mathcal{F}, \mathbb{P})$-measurable random variable and define
\[ M_n = E^\xi \left( \frac{\xi}{B_n} \bigg| \mathcal{F}_n \right), \quad n = 0, \ldots, N. \]
It is clear that $(M_n, \mathcal{F}_n, \mathbb{P})_{0 \leq n \leq N}$ is a martingale and therefore there exist $\mathcal{F}$-measurable functions $\gamma_n$, \( n = 1, \ldots, N, \) with
\[ M_n = M_0 + \sum_{k=1}^{n} \gamma_k \left( \frac{S_k}{B_k} - \frac{S_{k-1}}{B_{k-1}} \right), \quad n = 1, \ldots, N. \]
Now define
\[ \beta_n := M_n - \gamma_0 \frac{S_n}{B_n}, \quad n = 1, \ldots, N, \]
and
\[ \pi := \left\{ \{\beta_n, \gamma_n\} : n \geq 0 \right\}. \]
Due to the definition of the $\beta_n$’s the equalities $M_n = M_0$ hold for $n = 0, \ldots, N$ and hence $X_N = B_N M_N = B_N \cdot M_N = \xi$ is valid as well. Finally we must show that $\pi$ is self-financing. Indeed,
\[
\begin{align*}
B_{n-1} \Delta \beta_n + S_{n-1} \Delta \gamma_n &= B_{n-1} \left( \Delta M_n - \Delta \left( \frac{S_n}{B_n} \right) \right) + S_{n-1} \Delta \gamma_n \\
&= B_{n-1} \left( \gamma_n \Delta \left( \frac{S_n}{B_n} \right) - \Delta \left( \gamma_n \frac{S_n}{B_n} \right) \right) + S_{n-1} \Delta \gamma_n \\
&= -B_{n-1} \frac{S_{n-1}}{B_{n-1}} \Delta \gamma_n + S_{n-1} \Delta \gamma_n = 0.
\end{align*}
\]

Lemma 35 (Martingale Representation) In a d.t.b.-$(B, S)$ market each martingale $(M_n, \mathcal{F}_n, \mathbb{P})_{0 \leq n \leq N}$ with $E M_N = 0$ can be written in the form
\[ M_n = \sum_{k=1}^{n} \alpha_k \Delta m_k^* \quad \text{for} \quad 1 \leq n \leq N \]
with appropriate $\mathcal{F}_{n-1}$-measurable functions $\alpha_n$, \( 1 \leq n \leq N, \) where
\[ m_n^* := \sum_{k=1}^{n} (p_k - \gamma_k), \quad 1 \leq n \leq N. \]

\begin{proof}
The $\mathcal{F}_n$-measurability of $M_n$ \((n = 1, \ldots, N)\) implies the existence of Borel functions $\alpha_n : \mathbb{R}^n \rightarrow \mathbb{R}$ \((n = 1, \ldots, N)\) such that
\[ M_n(\omega) = \alpha_n(\rho_1(\omega), \ldots, \rho_n(\omega)), \quad \omega \in \Omega. \]
Therefore the martingale property of $M_n$ can be written in the form
\[ \mathbb{E}(h_n(\rho_1(\omega), \ldots, \rho_n(\omega)) \mid \mathcal{F}_{n-1}) = 0 \quad (\mathbb{P}^n \text{-a.s.}) \]
or equivalently
\[ (1 - p) h_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), b) + (1 - p) h_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), a) = h_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega)) \quad (\mathbb{P}^n \text{-a.s.}). \tag{9} \]
Now define
\[ \alpha_n := h_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), b_n) - h_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega)). \tag{10} \]
It can be easily derived from equation (9) that
\[ \alpha_n = h_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), a_n) - h_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega)). \tag{11} \]
Multiplying equations (10) and (11) by $(b_n - r_s)$ and $(a_n - r_s)$ respectively, one can notice that these two equations are equivalent to
\[ \Delta M_n(\omega) = \alpha_n(\omega) \Delta m_n^*, \quad n = 2, \ldots, N, \]
which was the statement to be proved. \qed
\end{proof}

Theorem 36 If the sequences of the interest rates and the coefficients of a d.t.b.-$(B, S)$ market satisfy the conditions
\[ \alpha_n < r_n < b_n \quad \text{for} \quad n = 1, \ldots, N \]
then the market is complete.

\begin{proof}
Let $(M_n, \mathcal{F}_n, \mathbb{P})_{0 \leq n \leq N}$ be a martingale where $\mathbb{P}$ is the equivalent martingale measure in the market defined in Theorem 27. By the application of Lemma 35 we get the form
\[ M_n - M_0 = \sum_{k=1}^{n} \alpha_k \Delta m_k^*, \quad 1 \leq n \leq N, \]
with appropriate random variables $\alpha_1, \ldots, \alpha_N$, which are $\mathcal{F}_1, \ldots, \mathcal{F}_N$-measurable respectively, where $m_n^*$ is the same as in the lemma. Then define
\[ \gamma_n := \alpha_n \frac{B_n}{S_n}, \quad n = 1, \ldots, N, \]

It is obvious that the $\gamma_n$'s are $\mathcal{F}_n$-measurable and

$$M_n = M_0 + \sum_{k=1}^{n} a_k \Delta m_k = M_1 + \sum_{k=1}^{n} \gamma_k \frac{S_{k-1}}{B_k} (p_k - r_k)$$

$$= M_0 + \sum_{k=1}^{n} \gamma_k \frac{S_{k-1}}{B_k} \left(\frac{p_k + 1}{r_k + 1} - 1\right) = M_0 + \sum_{k=1}^{n} \gamma_k \Delta m_k.$$ 

It follows from Theorem 27 and Theorem 34 that the market is complete. \qed

6 Options

6.1 Options as derivative securities

Consider a security market described in Section Security Markets, where the types of the basic underlying securities to be dealt with have been described. These securities can be used in order to construct more complicated financial contracts. In this way we obtain new financial securities called derivative securities, emphasizing their dependence on other (already existing) securities. Derivative security contracts contain certain rights of the parties with respect to the more basic financial assets and therefore their values depend on the values of those basic assets. That is why sometimes they are referred to as contingent claims as well. Forward and future contracts are examples of derivative securities, as well as an option contract which is the subject of our further investigation.

Our focus is mainly on the so-called European and American options which are the most common and well-known kinds of options. In both cases we can classify them by distinguishing between two basic types, one labelled as call and the other as put.

A call option contract on a certain asset gives the right to its holder (or buyer or owner) to buy one unit of the underlying asset from the option seller (or emitent) for a certain fixed price by a certain time (or date), whereas a put option gives its holder the right to sell the underlying asset for a fixed price by a certain time. The fixed price in the contract is called exercise price or strike (price) and denoted by $K$ in this paper. The final date by which the option can be exercised (i.e. the holder can buy/sell the asset according to the contracting conditions) is named to be the expiration date/time or maturity or terminal date in the literature, and we denote it by $T$ since it is the same time point until which we observe the market according to our interest.

The European and American options differ in the contracting terms regarding to the expiration date. A European option can be exercised only at maturity ($T$) in contrast to an American one which can be exercised at any time up to maturity.

It should be emphasized that option’s holder is not obligated to exercise his right written in the contract, it is his own decision to do so or not.

6.2 Exercising the option

Now let us consider a European call option on a stock in a security market. We denote the price of the underlying stock at time $t \in [0, T]$ by $S_t$, where 0 is the current date. We mention that we do not need to be restricted to the discrete time case for the forthcoming general remarks. Therefore we will use the notations $S_t, S_{B_t}, B_t, \ldots, B_T$ instead of $B_k, S_k$ for $t \in [0, T]$ according to Definition 1 only in case the corresponding remark is about the discrete time model.
If at time $T$ the stock price $S_T$ is larger than the exercise price then the holder can buy one share of stock at price $K$ and then sell it immediately in the market at price $S_T$ for the profit $S_T - K$. On the other hand, $S_T$ might be below or equal to $K$ which makes the contract worthless since it would make no sense to use the right given by the option if the stock is available at a lower price in the market. Thus at maturity time the option entitles its owner to the payment

$$(S_T - K)^+ := \max(0, S_T - K)$$

which is equal to the loss of the option seller obviously. In other words, the option is worth $(S_T - K)^+$ at time $T$.

Therefore one can find bounds for $K$ in an easy way. The value of $K$ should be taken from the range of the possible values of the stock price $S_T$, but this statement makes sense only if that range can be "predicted". For instance, in a d.t.h.b.-(B, $S$)$_N$ market with coefficients $a$ and $b$ the inequality

$$\min(S_X(\omega)) - S_0(1 + a)^N < K < S_0(1 + b)^N - \max(S_X(\omega))$$

should hold provided that the parties of the contract make rational economic decisions.

Arguments similar to the one above can be made about a European put option and also about American options with certain modifications. We mention just two of them. Unlike the holder of a call option the holder of a put option would like the stock price to go below the exercise price since in that case he or she could have his share bought by the emittent for more than the market price. In case of the American type it is obvious that $T$ and $S_T$ can be replaced by any time $t \in [0, T]$ and the corresponding price $S_t$ respectively.

### 6.3 Positions

According to the role one plays on a certain side of a contract we can define different positions. The buyer of the option is said to have taken the long (call or put) position whereas the option seller possesses the short (call or put) position by having potential liabilities in the future.

In a similar way, the ownership on a stock is called long stock position in contrast to short stock position which is an obligation to deliver a share of stock at maturity due to a short sale of stock.

A useful way of characterizing the options is to determine the payoff of the different positions when the options are exercised. Moreover, this is actually the only information about the option that we need in our mathematical model. The function showing the payoff received by the buyer of a certain option (when the option is exercised) will be called the corresponding payoff function of the option. It is easy to see that the payoff is equal to

- $(S_T - K)^+ := \max(S_T - K, 0) \geq 0$ for a European long call position and so
- $-\max(S_T - K, 0) = -\min(K - S_T, 0) \leq 0$ for a European short call position,
- $(K - S_T)^+ := \max(K - S_T, 0) \leq 0$ for a European long put position and so
- $-\max(K - S_T, 0) = -\min(S_T - K, 0) \leq 0$ for a European short put position.

Substituting $T$ by the time point $t \in [0, T]$ in the formulas above we get the possible corresponding payoffs of the different American option positions at time $t$ (if the option were exercised at $t$).

To get the total profits or losses from the positions (see Figure 3) we have to adjust the formulas above by including the price paid for the option at time $0$ by the holder (and received by the emittent). Obviously it just means that we add or substitute the price of the option which is expressed in terms of the expiration date. Here we mention that generally in economics and especially in our theory, calculating different figures on the same base by the use of the discount factor (e.g., derived from the interest rate) is essential in order to make the figures comparable. In Figure 3 we chose the terminal date as the base for calculating the total profits but it could have been any other time point in $[0, T]$. Therefore each price paid for the option at time $0$ is multiplied by the reciprocal of the discount factor (i.e., by $B_T/B_0$) which is, e.g., equal to $\prod_{s=1}^{N} (1 + n_s)$ in a binary market described in Section 2.

The emittent of a call option can organize a so-called covered call position by buying a stock at the same time when the option is sold. Therefore the emittent’s possible loss $((S_T - K)^+)$ would be covered if the option were exercised. Similarly, a covered put position means owning a stock share and a put option. Having a look at Figure 3 it is easy to check that the following two equivalences are valid (in terms of total profit):

$$\text{covered call (long stock + short call)} \iff \text{short put}. \quad (12)$$

$$\text{covered put (long stock + long put)} \iff \text{long call}. \quad (13)$$

These simple relations among the different positions played an important role especially in the early years of the history of options when the trading of put options was not allowed but “artificial put positions” were possible to organize according to (12) and (13).

### 6.4 The basic problem: pricing the option

We have described the main types of options and derived their payoff functions which correspond to the maturity. In other words, one can say that the payoff function tells us what the option contract is worth at the expiration date.
In the remaining part of this section the price of the certain investigated call (put) option will be denoted by $C$ ($P$), whereas the indices $Eu$ and $Am$ are to indicate whether they are of European or American type.

The basic problem concerning options is the following. “How much would you be willing to pay at the initial time point $0$ for the option, which assures you the right to receive a (random) payment at maturity?” To put it in another way, “What is the fair or rational price of the option at the initial time $0$?”

Consider a European call option with strike price $K$ and maturity $T$, which, as we saw, gives the holder the right to buy one share of stock at time $T$ for price $K$. To approach the problem of option pricing, first we should characterize what fairness of the price naturally could be meant, that is what could be a fair price both from the point of view of the seller and from the point of view of the buyer. Notice that at time $0$ the emitten gets a payment, namely, the price for the option. On the other hand he contracts an obligation which is actually equivalent to undertaking to pay $(S_T - K)^+$ to the buyer at $T$. Therefore, in terms of fairness, the rational price of the option should be enough for making the seller have the opportunity to cover his or her possible losses (i.e., $(S_T - K)^+$ at time $T$.

Hence we turn out to find reasonable to call $C_{f_{N,f_{E}}}$ (see Definition 18) the fair price of the option where $f_{N}(S_N) = (S_N - K)^+$. This conclusion is based on the following facts.

- This is a capital which assures the emitten the opportunity to organize a hedging strategy against the desired wealth $(S_N - K)^+$ and it is the minimal of such kind of initial capitals (see Remark 20) which means that
- in case of any lower price the option seller would not be able to satisfy the contract conditions and finally,
- any higher price $C$ would cause the existence of arbitrage because it would give the seller the opportunity to realize immediately the riskless profit $C - C_{N,f_{E}}$ since $C_{N,f_{E}}$ would be enough to discharge the contracting obligation by using the same hedging strategy (with initial capital $C_{N,f_{E}}$ against $(S_N - K)^+$ as described above.

In a similar way, the fair price of any European option can be obtained by letting $f_{N}$ be equal to the appropriate payoff function of the option.

Now let us consider generally a contingent claim whose payoff can be written in the form $f_{N}(S_0, S_1, \ldots, S_N)$. It means that the payment might depend on the whole history of the market. Then the way we followed above is suitable to price any of such kind of contingent claim as well.

We will see later that any other option price, which differs from the fair price just been characterized, would cause arbitrage opportunity either for the option seller or the buyer.

We have two important open questions about the problem of pricing the options. One is to calculate the fair price and derive formulas which can be used in practice. The second is to
show the existence of at least one self-financing strategy whose value process starts with the
fair price of the option as its initial capital \((X^T_0)\) and ends with the payoff function value
of the option as its capital at maturity \((X^T_0)\). Obviously the fair price formulas are
functions of several variables given either by the contracting conditions or by parameters of
the market: the strike \(K\), the expiration date \(T\), the initial bond and stock prices \(B_0\) and
\(S_0\), the parameters of the stochastic processes representing the stock price evolution and
the bond’s interest rate (which determines the bond price evolution or vice versa).

6.5 Call versus Put

Now suppose that the options are traded at their fair price in the market. Then one can
derive direct relations between the call option and put option prices. Here we mention only
some of them.

In case of the European type of options [with the same contract parameters] the equation
\[ C^E_0 + K \frac{B_0}{B_T} - S_0 = P^E_0 \]
describes the relation which is known as the put-call parity. To verify it one can imagine a
portfolio \(I\) consisting of one European call option plus \(K B_0 / B_T\) units of bond and a portfolio
\(I_1\) consisting of one European put option plus one share of stock. (Here we mention that on
the analogy of portfolio defined on two securities in Section 2 one can define portfolio on \(n\)
securities generally as it is common in economics.) Then portfolio \(I\) is worth
\[ (S_T - K)^+ + K = \max(S_T, K) \]
at time \(T\) as well as portfolio \(I_1\) since
\[ (K - S_T)^+ + S_T = \max(K, S_T). \]
Therefore these portfolios should have identical values at any time point in \([0, T]\), particularly
\[ C^E_0 + K \frac{B_0}{B_T} = P^E_0 + S_0 \]
should be also valid.

In case of American options certain bounds can be derived for the difference of the prices.
We are not going to deal with this [see [Hull 93]].

6.6 Options in reality

Finally we mention that the notion option has been common throughout the world since
1973 when they were traded first time on an organized exchange. That was the year of the
great breakthrough in the option pricing problem as well since F. Black and M. Scholes
derived their famous and celebrated pricing result which is known as Black-Scholes formula
([Black and Scholes 73]). Here we mention that it refers to the continuous time model.

There are several stock exchanges where options are traded like the Chicago Board Options
Exchange (CBOE) or the Philadelphia Exchange (PHLX) and there are plenty of different
kinds of option contracts which can differ in the exercise conditions or in the definition of
the strike or in the underlying asset which they are defined on. Therefore we just mention a
few further types of options to illustrate the big range of options. At the same time we stress
again that option contracts can be defined on any risky asset which matches the definition
of stock in the theory and it is also important to note that the forthcoming pricing formulas
are suitable to price a wide range of contingent claims.

Regarding to the underlying asset, options on stock, options on forward contracts, options on
foreign currency are quite ordinary nowadays, but one can imagine options even on options.
The following examples are meant to be used in discrete time model and therefore we turn
again to the notations \(B_n\) and \(S_n\) for \(n = 1, \ldots, N\) (see Definition 6).

The look back option which is a derivative security contracted on the analogy to a European
option except for the fact that the strike is replaced in this case by a function of the history
of the stock till maturity, namely, its payoff function has the following form:
\[ (S_N - K_N)^+ \quad \text{with} \quad K_N := \min(S_0, S_1, \ldots, S_N) \]
for the look back call option and
\[ (K_N - S_N)^+ \quad \text{with} \quad K_N := \max(S_0, S_1, \ldots, S_N) \]
for the look back put option.

A further well-known option type is the Asian one where the strike and therefore the payment
is defined in terms of the average value of the stock with respect to the period \([0, T]\). More
precisely, the call and put payoff functions are again of the form \((S_N - \bar{K})^+\) and
\((\bar{K} - S_N)^+\) respectively but with
\[ \bar{K} := \frac{1}{N+1} \sum_{i=0}^{N} S_i \]
Finally we mention the Bermudian option which can be exercised only on certain specified
days during its life \([0, T]\).

For more about option types and economic notes on them see [Hull 93].
7 Pricing European Options

7.1 The basic statements

The statement of the following lemma is fairly trivial, however we find it important to stress because it explains the role played by an equivalent martingale measure in option theory.

Lemma 37 Let the self financing strategy \( \pi \) be an \((x, f_N)\)-hedge in a d.f.-\((B, S)\) market where \( x \in \mathbb{R}^n \) and \( f_N : \mathbb{R}^{n+1} \to \mathbb{R} \) is a Borel function and assume that \( \mathbb{P}^* \) is an equivalent martingale measure in the market.

Then
\[
x \geq \frac{B_0}{B_N} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N)
\]
and if additionally \( \pi \) is a minimal \((x, f_N)\)-hedge then
\[
x = \frac{B_0}{B_N} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N),
\]
where \( \mathbb{E}^* \) denotes the expected value with respect to \( \mathbb{P}^* \).

Proof. By the martingale property of \((M^*_N, \mathcal{F}_n, \mathbb{P}^*)_{0 \leq n \leq N} \), we have
\[
\frac{B_0}{B_N} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N) \leq \frac{B_0}{B_N} \mathbb{E}^* X^*_N - B_1 \mathbb{E}^* X^*_1 - \ldots - B_N \mathbb{E}^* X^*_N - x
\]
\[= B_0 \mathbb{E}^* M^*_N - B_1 M^*_1 - \ldots - B_N M^*_N - x \tag*{\Box}
\]

Theorem 38 (Price of Contingent Claim) Let \( \mathbb{P}^* \) denote the (unique) equivalent martingale measure in a complete d.f.-\((B, S)\) market and suppose that \( f_N : \mathbb{R}^{n+1} \to \mathbb{R} \) is a Borel function.

Then
\[
C_{N, f_N} = \frac{B_0}{B_N} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N).
\]

Proof. By Lemma 19 we know that there is an \( x \in \mathbb{R}^n \) such that \( \Pi(\cdot, f_N) \neq \emptyset \) which means that \( C_{N, f_N} < \infty \). By Lemma 37 it is obvious that
\[
C_{N, f_N} \geq \frac{B_0}{B_N} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N).
\]
Furthermore the completeness of the market assures us the existence of a self-financing minimal \((x, f_N)\)-hedge \( \pi \) for which
\[
X^*_\pi = \frac{B_0}{B_N} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N)
\]
and hence \( X^*_\pi \geq C_{N, f_N} \), that is, in inequality (14) the equality is satisfied as well. \( \Box \)

Remark 39 It can easily be seen from the proof of implication (3) \( \Rightarrow \) (1) in Theorem 34 that the inequality \( f_N(S_0, S_1, \ldots, S_N) \geq 0 \) implies immediately the existence of a minimal \((C_{N, f_N}, f_N)\)-hedge strategy with non-negative value process \( (X^*_\pi)^{\mathcal{F}_n} \geq 0 \) on the time interval \([0, T]\).

Next we summarize the results given by Lemma 37 and Theorem 38 in order to get the basic theorem of pricing the European type of options. The adjective European refers to the fact that the option can be exercised only at maturity but it need not necessarily have one of the known payoffs like \((S_0 - K)^+\) or \((K - S_0)^+\).

7.2 Some commonly used pricing formulas

Theorem 40 (Fair Price of European Option) Suppose that the inequalities \( a_n < r_n < b_n \) hold for \( n = 1, \ldots, N \) in a d.f.-\((B, S)\) market with interest rates \( \{r_n\}_{n=1}^N \) and coefficients \( \{a_n\}_{n=1}^N \) and \( \{b_n\}_{n=1}^N \) and let \( f_n : \mathbb{R}^{n+1} \to \mathbb{R} \) be a Borel function.

1. The fair price of a European type option with payoff function \( f_N(S_0, S_1, \ldots, S_N) \) is
\[
C_{N, f_N} = \frac{1}{\prod_{n=1}^N (1 + r_n)} \mathbb{E}^* f_N(S_0, S_1, \ldots, S_N)
\]
where \( \mathbb{E}^* \) denotes the expected value with respect to the probability measure \( \mathbb{P}^* \) defined by
\[
\mathbb{P}^*(p_n = b_n) = \frac{b_n - a_n}{b_n - a_n}, \quad n = 1, \ldots, N.
\]

2. There also exists a self-financing strategy \( \pi \) which is a minimal \((C_{N, f_N}, f_N)\)-hedge and one of such kind of strategies at issue can be given explicitly by the formulas
\[
\pi := \{\pi_n = (b_n - a_n) \gamma_n\}_{n=0}^N
\]
where
\[
\gamma_n := \frac{a_{n+1}}{b_n} - a_n, \quad n = 1, \ldots, N
\]
and \( b_n \)'s \( (n = 1, \ldots, N) \) are the \( \mathcal{F}_n \)-measurable coefficient functions in the martingale representation of \( X^*_\pi/B_n \) given by Lemma 35.
Proof. Statement (1) follows from Theorem 38, whereas statements (2) and (3) follow from Theorem 36.

Corollary 41 Suppose that the conditions of Theorem 40 are satisfied. Let us assume that \( f_N \) can be written in the form

\[
g(S_N(\omega)) = f_N(S_0(\omega), S_1(\omega), \ldots, S_N(\omega)), \quad \forall \omega \in \Omega,
\]

with appropriate Borel function \( g : \mathbb{R} \to \mathbb{R} \).

Then

\[
C_{N,f_N} = \frac{1}{\prod_{k=1}^N (1 + r_k)} \sum_{H \subseteq \{1, \ldots, N\}} \prod_{k \in H} S_k \prod_{k \notin H} (1 + a_k) \prod_{k \in H} \rho_k^n \prod_{k \notin H} (1 - \rho_k^n)
\]

where the notations are the same like in Theorem 40 and \( \Gamma \) is the power set\(^2\) of the set \( \{1, 2, \ldots, N\} \). Particularly,

\[
C_{N,f_N} = \frac{1}{(1+r)^N} \sum_{k=1}^N g \left( \frac{S_0(1+b)^k(1+a)^{N-k}}{N} \right) \left( \frac{N}{k} \right) (p^*)^k (1-p^*)^{N-k}
\]

in a homogeneous binary market where \( p^* := (r-a)/(b-a) \).

Proof. It is clear that formulas (15) and (16) are exactly

\[
\prod_{k=1}^N (1 + r_k) \mathbb{E} g(S_N) \quad \text{and} \quad (1 + r)^{-N} \mathbb{E} g(S_N)
\]

respectively.

Corollary 42 (Cox-Ross-Rubinstein Pricing Formula) In a \( d \times \ell \times b \times (B, S)_N \) market the fair price of a European call-option with exercise price \( K \) \((K > 0)\) and with payoff function \( (S_N - K)^+ \) is

\[
C_{N,K}^{\text{call}} = S_0 \mathbb{B}(k_0, N, \tilde{p}) - K(1+r)^{-N} \mathbb{B}(k_0, N, p^*)
\]

where

\[
k_0 := 1 + \left\lfloor \frac{\log \left( \frac{S_0(1+a)^N}{1+b} \right)}{\log \left( 1 + \frac{b-a}{1+a} \right)} \right\rfloor
\]

\(^2\) \( \Gamma \) is the collection of all subsets of the set \( \{1, 2, \ldots, N\} \).

and

\[
\mathbb{B}(j, N, p) := \begin{cases} \sum_{k=j}^N \binom{N}{k} p^k (1-p)^{N-k} & \text{if } k \leq N, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}
\]

\[
\tilde{p} = \frac{1+b}{1+a}, \quad p^* = \frac{r-a}{b-a}
\]

(where \( |y| \) \((y \in \mathbb{R})\) denotes the largest integer which is smaller than or equal to \( y \)).

Proof. Let us choose \( g(x) = \max(0, x-K) \) and apply Corollary 41.

Suppose that an event, say, \( \tilde{\omega} \in \Omega \) has been realized in the market the realization \((\rho_1(\tilde{\omega}), \ldots, \rho_N(\tilde{\omega}))\) of which contains \( k \) \((0 \leq k \leq N)\) times the element \( 1+b \) that is, during the observed time interval upward jump happened to the stock price strictly \( k \) times. Then

\[
g(S_N(\tilde{\omega})) = \max \left( 0, S_0(1+a)^N \left( \frac{1+b}{1+a} \right)^k - K \right)
\]

furthermore,

\[
S_0(1+a)^N \left( \frac{1+b}{1+a} \right)^k - K > 0
\]

if and only if

\[
k > \log \frac{K}{\log \left( \frac{S_0(1+a)^N}{1+b} \right)} \frac{1+b}{1+a}
\]

that is, if \( k \geq k_0 \). We get

\[
C_{N,K}^{\text{call}} := C_{N,f_N} = S_0 \sum_{k=k_0}^N \binom{N}{k} p^k (1-p)^{N-k} \left( \frac{1+b}{1+r} \right)^N \left( \frac{1+b}{1+a} \right)^k
\]

\[
- K(1+r)^{-N} \sum_{k=k_0}^N \binom{N}{k} p^k (1-p)^{N-k}
\]

\[
= S_0 \mathbb{B}(k_0, N, \tilde{p}) - K(1+r)^{-N} \mathbb{B}(k_0, N, p^*)
\]

since

\[
1 - \tilde{p} = 1 - \frac{1+b}{1+r} \frac{(b-a)(1+r) - (r-a)(1+b)}{(b-a)(1+r)} = - \frac{(b-r)(1+a)}{(b-a)(1+r)} = (1-p^*) \frac{1+a}{1+r}
\]

\( \blacksquare \)
Corollary 43 (Put-Call Parity) The fair price of a European put option with exercise price $K$ ($K > 0$) and with payoff function $(K - S_N)^+$ in a d.t.b.-$(B, S)_X$ market is

$$C_{N,K}^{put} = C_{N,K}^{call} - S_0 + \frac{K}{\prod_{n=1}^{N}(1 + r_n)}$$

where $C_{N,K}^{call}$ is the corresponding call option price (with the same strike).

Proof:

$$C_{N,K}^{put} := \frac{1}{\prod_{n=1}^{N}(1 + r_n)} \mathbb{E}^p \max(0, K - S_N) = \frac{1}{\prod_{n=1}^{N}(1 + r_n)} \mathbb{E}^p [\max(S_N - K, 0) - S_N + K]$$

$$= C_{N,K}^{call} - \frac{1}{\prod_{n=1}^{N}(1 + r_n)} \mathbb{E}^p S_N + K(1 + r)^{-N} = C_{N,K}^{call} - S_0 + \frac{K}{\prod_{n=1}^{N}(1 + r_n)},$$

since, by the independence of $\rho_1, \ldots, \rho_N$, $\mathbb{E}^p S_N = S_0 \prod_{n=1}^{N}(1 + r_n)$ where $\mathbb{P}^*$ is the unique equivalent martingale measure in the market. \qed

7.3 Example (pricing in practice)

In this example we demonstrate an application of our results obtained above. Consider an option on foreign currency in a discrete time $(B, S)_X$ market. Suppose that the prices are given in terms of German Mark (DM), the interest rate is 5% and the underlying foreign currency is the US dollar with exchange rate 1.7.

Let us imagine a European call option on 100 US $ with exercise price 170 DM where the contract can be exercised after 1 year. Let us also assume that the dollar exchange rate can become 1.53 or 2.21 (after one year), that is the value of the dollar measured in DM can decrease by 10% or increase by 30% respectively.

In this case the bond is a bank account while one share of stock is 100 US $. Reformulating our assumption, we have (see Definition 6)

$$B_0 = 1, \quad r_t = 0.05, \quad B_t = 1.05$$

$$S_0 = 170, \quad \text{and} \quad S_t = (1 + r_t)S_0$$

where $p_t$ can take either $a_1 = -0.1$ or $b_t = 0.3$ at maturity. Furthermore, $K = 170$ and

$$p' - \mathbb{P}^* (p_t - b_t) = \frac{r_t - a_1}{b_t - a_1} = 0.05 + 0.1 = 0.35 \cdot 0.30 = 0.335.$$  

The payoff function has the form $\max(S_t - K, 0)$ and hence the fair price of the option is (see Corollary 41 or Theorem 40)

$$C = \frac{1}{1.05} \left( (1 - p^*) + 0 \cdot (1 - p^*) \right) = 51 \cdot 0.375 = 18.21.$$  

We can also construct the strategy (due to Theorem 40) that the emitten should organize in order to satisfy the contracting conditions. By equation (10) in Lemma 35 we get

$$\alpha_1 := \frac{51/1.05 - 18.21}{0.3 - 0.05} = 121.42$$

or equivalently, by equation (11) in the same lemma

$$\alpha_1 = \frac{0 - 18.21}{-0.1 - 0.05} = 121.42,$$

and thus Theorem 40 says that

$$\gamma_1 := \frac{\alpha_1 B_1}{S_1} = 0.75.$$

$$\beta_1 := \frac{C - \gamma_1 S_1}{B_1} = -109.28.$$
\[ \pi_0 := (C, 0), \quad \pi_1 := (-109.28, 0.75). \]

The emittent gets capital (price) \( C \) for the option at time \( 0 \) and then according to the strategy defined above he should take on a loan (equal to 109.28 DM) and thus he can possess 109.28 + 18.21 DM for what he should buy 75 US dollars.

At time \( T \) the emittent’s debt is \( 109.28 \cdot 1.05 - 114.75 \) DM that he must pay back to the bank. There are two possibilities. If the dollar exchange rate has increased \( (\rho_1 = b_1) \) then the emittent’s capital is

\[ X_T^\pi = -114.75 + 0.75 \cdot 221 - 51, \]

i.e., after returning the debt back to the bank, he has got 51 DM which is precisely the amount of money he must pay to the buyer of the option. If the dollar exchange rate has decreased \( (\rho_1 = a_1) \), the value of the dollars possessed by him will cover precisely his debt, since

\[ X_T^\pi = -114.75 + 0.75 \cdot 153 = 0, \]

and obviously the option will not be exercised in this case.

So, we have seen that the emittent will be able to discharge his obligation in both cases indeed.

8 The Continuous Setting

So far we studied the problem of contingent claim valuation and related topics in discrete time models. Now we turn to continuous time models where we will build up a mathematical setting on the analogy of the discrete case to investigate and to answer to the same problems. We will try to keep the terminology and the structure we used so far as much as it is possible and thus explanations of the forthcoming notions will be given only in the case of major differences. However, we mention that the continuous extension or version of our means sometimes requires some additional technical conditions to make them well defined at all.

8.1 The continuous markets

The qualifier ‘continuous time’ here refers to the fact that, given again a time interval \([0, T]\) with current date 0 and terminal date \( T \) in the future \((T \in \mathbb{R}^+)\), each time \( t \) in \([0, T]\) is a trading time. This means that prices might change any time until \( T \) and tradings are also allowed at each \( t \in [0, T] \). Thus all the processes (like the price, value or the portfolio processes) are defined over \([0, T]\).

We keep the main assumptions on the market: two available securities (the deterministic bond and the random stock) are traded, there are no transaction costs, borrowing and lending at the risk-free interest rate is allowed. The symbols \( B, S, \beta, \gamma, \pi, X^\pi, M^\pi \), are to denote the same prices and assets as before and the subscript \( t \) \((t \in [0, T]) \) will show the time parameter of these processes.

Now one can put the question: “How can the self-financing property be characterized in continuous trading?”. For this, first imagine an investor who adjusts his portfolio only \( k \) times until \( T \), say, at \( 0 < t_1 < t_2 < \ldots < t_k < T \) and set \( t_0 := 0, t_{k+1} := T \). Then his strategy \( \pi := \{\pi_t = (\beta_t, \gamma_t) \mid t \in [0, T]\} \) satisfies

\[ \beta_t = \sum_{i=0}^{k} 1_{(t_{i+1}, t_i]}(t) \beta_{t_i} \quad \text{and} \quad \gamma_t = \sum_{i=0}^{k} 1_{(t_{i+1}, t_i]}(t) \gamma_{t_i}, \]

and hence his capital gain over \([0, T]\) is

\[ \sum_{i=0}^{k} (B_{t_{i+1}} - B_{t_i}) \beta_{t_i} + \sum_{i=0}^{k} (S_{t_{i+1}} - S_{t_i}) \gamma_{t_i} - \int_0^T \beta_t dB_t - \int_0^T \gamma_t dS_t. \]

Therefore we will require a self-financing strategy \( \pi \) to satisfy

\[ X_T^\pi = \beta_T B_T + \gamma_T S_T = X_0^\pi + \int_0^T \beta_t dB_t + \int_0^T \gamma_t dS_t, \]

where certain assumptions will make the stochastic integrals above well-defined.
Definition 44 (Continuous Time \((B,S)_T\) Market) The set 
\[ \{\Omega, \mathcal{F}, \mathbb{P}=(\mathbb{F}_t) \}, B=(B_t), S=\{S_t\}_T \] is called continuous time \((B,S)_T\) market if

- \(T \in \mathbb{R}^+\),
- \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space with a standard filtration \(\mathbb{F} = \{\mathcal{F}_t : t \in [0,T]\}\),
- \(B = \{B_t : t \in [0,T]\}\) called the bond price process, is a strictly positive continuous deterministic process and has bounded variation over \([0,T]\),
- \(S = \{S_t : t \in [0,T]\}\), called the stock price process, is a strictly positive \(\mathbb{R}CLL\) process adapted to \(\mathbb{F}\).

Definition 45 (Black-Scholes Market) We call a continuous time \((B,S)_T\) market 
\[ \{\Omega, \mathcal{F}, \mathbb{P}=(\mathbb{F}_t) \}, B=(B_t), S=\{S_t\}_T \] a Black-Scholes market if

- \(B_t = B_0 e^{\mu t}\) for \(t \in [0,T]\) where \(B_0 > 0\) and \(\mu \geq 0\),
- the stock price process \(S = \{S_t : t \in [0,T]\}\) is determined by the stochastic integral equation
  \[ S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s, \quad t \in [0,T] \] (17)
  where \(W = \{W_t : t \in [0,T]\}\) is a standard Brownian motion (or Wiener process) in 
  \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) such that \(\mathbb{F}\) is the standard filtration generated by \(W\), that is, 
  \(\mathcal{F}_t = \sigma\{\sigma(W_s : 0 \leq s \leq t) \cup A \in \mathcal{F} : P(A) = 0\}\), and
  - \(\mu \in \mathbb{R}, \sigma > 0, S_0 > 0\) are constants.

Remark 46 The stochastic integral equation (17) can be written in the formal differential form

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0,T], \] (18)

but the reader should keep in mind that the rigorous definition of equation (18) is equation (17).

Then a strong solution of (17) is a continuous process \(S = \{S_t : t \in [0,T]\}\) such that

- it is adapted to the filtration \(\mathbb{F}\),
- \(\int_0^t S_s^2 ds < \infty \mathbb{P}\text{-a.s. for each } t \in [0,T] \), and
- \(S\) satisfies equation (17) \(\mathbb{P}\text{-a.s. for each } t \in [0,T] \).

Now we show that the process

\[ S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right), \quad t \in [0,T], \]
is a unique solution of (17). Indeed, by Itô's formula (see VI.39., page 394 in [Rogers and Williams]) we have

\[ S_t - S_0 = \int_0^t \sigma S_u dW_u + \int_0^t \left( \mu - \frac{\sigma^2}{2} \right) S_u du + \frac{1}{2} \int_0^t \sigma^2 S_u d[W]_u, \]

\[ - \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u, \quad t \in [0,T], \]

where \([W]\) denotes the quadratic variation process of \(W\). The uniqueness follows immediately from the fact that the coefficient functions in (17) (namely, \(f_1(x) = \mu x\) and \(f_2(x) = \sigma x\)) are linear and thus they clearly satisfy a Lipschitz condition, see Section 10.2 in [Chung and Williams 90]. Here a strong solution is meant in the sense as it is discussed in [Chung and Williams 90] though we mention that other interpretation of the notion at issue, which are not necessary equivalent to the one we used, can also be found in the literature.

Definition 47 In a c.t.-\((B,S)_T\) market \(\mathbb{P}^*\) is called an equivalent martingale measure (EMM) if

- \(\mathbb{P}^*\) is a probability measure on \((\Omega, \mathcal{F})\),
- \(\mathbb{P}\) and \(\mathbb{P}^*\) are equivalent and
- the process \(\{S/B_t : t \in [0,T]\}\) forms a martingale with respect to \(\mathbb{P}^*\).

Notation 48 For convenience we shall omit the underlying data set \(\{\Omega, \mathcal{F}, \mathbb{P}=(\mathbb{F}_t) \}, B=(B_t), S=\{S_t\}_T\) of a continuous time \((B,S)_T\) market if it does not cause confusion and we shall simply denote such a market general by c.t.-\((B,S)_T\) and, particularly, by c.t.-\((B,S)_T\) if there exists an EMM on it. A Black-Scholes \((B,S)_T\) market will be meant to be one satisfying Definition 45. In this theory, all the measures we are going to deal with are equivalent to the market measure \(\mathbb{P}\) thus it will suffice to work a.s. instead of \(\mathbb{P}\text{-a.s. or } \mathbb{P}^*\text{-a.s.}, etc.

Remark 49 The integration-by-parts formula (see VI.38., pp. 391-392 in [Rogers and Williams])

\[ X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t, \quad t \in [0,T], \]

for \(\mathbb{R}CLL\) semimartingales \(X = \{X_t : t \in [0,T]\}\) and \(Y = \{Y_t : t \in [0,T]\}\) shows us that the product of two semimartingales is a semimartingale as well. Moreover, the semimartingale property is known to be invariant to the substitution of equivalent measures.
But \( B \) has bounded variation while \( S/B \) is a martingale with respect to an EMM \( \mathbb{P}^* \) in a c.t.(\( B, S \)) \( \mathbb{P} \) market. Thus the existence of an EMM implies that \( S \) is a semimartingale with respect to the market measure \( \mathbb{P} \). This fact makes easy to define the self-financing property of a trading strategy. Notice that the integrals in the self-financing condition (19) are well defined. One could certainly define the self-financing property without the use of an EMM but requiring some integrability conditions on the strategy. However, in the pricing problem we will always need the existence of an EMM and that is why we have chosen this way.

### 8.2 Strategies

**Definition 50** In a c.t.(\( B, S \)) \( \mathbb{P} \) market let the processes \( \beta = [\beta_t \mid t \in [0, T]] \), \( \gamma = [\gamma_t \mid t \in [0, T]] \) are predictable, RCLL with constants \( \beta_t, \gamma_t \in \mathbb{R} \). Then the vector process \( \pi = \{\pi_t := (\beta_t, \gamma_t) \mid t \in [0, T]\} \) is called a (trading) strategy with value process \( X^\pi = \{X^\pi_t := \beta_t B_t + \gamma_t S_t \mid t \in [0, T]\} \) and discounted value process \( M^\pi = \{M^\pi_t := \frac{S_t}{B_t} \mid t \in [0, T]\} \). The vector \( \pi_t := (\beta_t, \gamma_t) \) is called the portfolio at time \( t \) (\( t \in [0, T] \)) corresponding to strategy \( \pi \).

In addition, let us assume that there exists an equivalent martingale measure in the market. Then we say that a strategy \( \pi = \{\pi_t := (\beta_t, \gamma_t) \mid t \in [0, T]\} \) is self-financing if \( \beta \) and \( \gamma \) are locally of bounded variation, and \( \mathbb{P} \)-a.s. for each \( t \in [0, T] \)

\[
X^\pi_t = X^\pi_0 + \int_0^t \beta_s d\langle B \rangle_s + \int_0^t \gamma_s d\langle S \rangle_s
\]  

(19)

**Lemma 51** In a c.t.(\( B, S \)) \( \mathbb{P} \) market assume that the strategy \( \pi = \{\pi_t := (\beta_t, \gamma_t) \mid t \in [0, T]\} \) has entries entries \( \beta, \gamma \) with finite variation. Then the following are equivalent:

1. \( \pi \) is self-financing,
2. \( X^\pi_t = X^\pi_0 + \int_0^t \beta_s d\langle B \rangle_s + \int_0^t \gamma_s d\langle S \rangle_s, \quad t \in [0, T] \),
3. \( \int_0^t B_s \beta_s d\langle B \rangle_s + \int_0^t S_s \gamma_s d\langle S \rangle_s + \sum_{0 < s \leq t} \Delta \gamma_s \Delta S_s = 0, \quad t \in [0, T] \).

In particular, in case of continuous price process \( S \) (3) reduces to

4. \( \int_0^t B_s d\langle B \rangle_s + \int_0^t S_s d\langle S \rangle_s = 0, \quad t \in [0, T] \).

**Proof.** By the integration-by-parts formula
\[
B_t \beta_t - B_0 \beta_0 = \int_0^t \beta_s d\langle B \rangle_s + \int_0^t B_s \beta_s d\langle B \rangle_t + [\beta, B]_t, \quad t \in [0, T],
\]  

(20)

\[
S_t \gamma_t - S_0 \gamma_0 = \int_0^t \gamma_s d\langle S \rangle_s + \int_0^t S_s \gamma_s d\langle S \rangle_t + [\gamma, S]_t, \quad t \in [0, T],
\]  

(21)

with
\[
[\beta, B]_t = 0 \quad \text{and} \quad [\gamma, S]_t = \sum_{0 < s \leq t} \Delta \gamma_s \Delta S_s, \quad t \in [0, T],
\]

since \( \beta \) and \( \gamma \) are locally of bounded variation. From equations (20), (21) we have

\[
X^\pi_t - X^\pi_0 = \int_0^t \beta_s d\langle B \rangle_s + \int_0^t \gamma_s d\langle S \rangle_s + \int_0^t B_s d\beta_s + \int_0^t S_s d\gamma_s
\]

\[+ \sum_{0 \leq s \leq t} \Delta \beta_s \Delta B_s + \sum_{0 \leq s \leq t} \Delta \gamma_s \Delta S_s, \]

which combined with the self-financing condition (19) completes the proof. \( \square \)

**Lemma 52** In a c.t.(\( B, S \)) \( \mathbb{P} \) market the discounted value process of a self-financing strategy \( \pi = \{\pi_t := (\beta_t, \gamma_t) \mid t \in [0, T]\} \) admits the integral representation

\[
M^\pi_t = M^\pi_0 + \int_0^t \gamma_s d\frac{S_s}{B_s}, \quad t \in [0, T].
\]

(22)

**Proof.** Note that the continuity and finite variation of the bond price process implies for all \( t \in [0, T] \)

\[
\left[ \frac{1}{B}, X^\pi \right]_t = 0 \quad \text{and} \quad \left[ \frac{1}{B}, S \right]_t = 0.
\]

(23)

Thus we have for all \( t \in [0, T] \)

\[
\frac{S_t}{B_t} - \frac{S_0}{B_0} = \int_0^t \frac{1}{B_s} d\langle S \rangle_s + \int_0^t S_s d\frac{1}{B_s},
\]

(24)

\[
\frac{X^\pi_t}{B_t} - \frac{X^\pi_0}{B_0} = \int_0^t \frac{1}{B_s} dX^\pi_s + \int_0^t X^\pi_s d\frac{1}{B_s},
\]

(25)

Furthermore,

\[
X^\pi_t = \beta_t S_t + \gamma_t S_t \quad \text{for} \quad t \in [0, T].
\]

(26)

Finally, from (23), (24), (25), (26) we obtain

\[
M^\pi_t - M^\pi_0 = \frac{X^\pi_t}{B_t} - \frac{X^\pi_0}{B_0}
\]

\[
= \int_0^t \frac{1}{B_s} dX^\pi_s + \int_0^t X^\pi_s d\frac{1}{B_s} - \int_0^t \frac{1}{B_s} d(\langle X^\pi \rangle_t - X^\pi_t) + \int_0^t X^\pi_s d\frac{1}{B_s}
\]

\[
- \int_0^t \frac{1}{B_s} \beta_s d\langle B \rangle_s + \int_0^t \frac{1}{B_s} \gamma_s d\langle S \rangle_s + \int_0^t (\beta_s d\langle B \rangle_s + \gamma_s d\langle S \rangle_s) d\frac{1}{B_s}
\]

\[
= \int_0^t \frac{1}{B_s} \beta_s d\langle B \rangle_s + \int_0^t \frac{1}{B_s} \gamma_s d\langle S \rangle_s + \int_0^t \frac{1}{B_s} \gamma_s d\langle S \rangle_s + \int_0^t \gamma_s d\frac{1}{B_s}
\]

\[
- \int_0^t \beta_s d\langle B_s \frac{1}{B_s} \rangle + \int_0^t \frac{1}{B_s} \gamma_s d\frac{S_s}{B_s} \quad \text{for} \quad t \in [0, T].
\]

(27)
8.3 Hedging and pricing

**Definition 53** Suppose that $x \in \mathbb{R}$ and $\xi$ is a ($\mathcal{F}$-measurable) random variable in a c.i.-$(B, S)_T$ market. Then a strategy $\pi$ is called $(x, \xi)$-hedge (or hedging strategy) if

$$X_0^\pi = x \quad \text{and} \quad X_T^\pi \geq \xi \text{ a.s.}$$

If, furthermore, $X_T^\pi = \xi$ is satisfied then $\pi$ is said to be minimal. The set of all self-financing $(x, \xi)$-hedges is denoted by $\Pi(x, \xi)$ whereas

$$C_{\xi, \xi} := \inf\{x > 0 \mid \Pi(x, \xi) \neq \emptyset\}$$

is called the investment cost (price), guaranteeing at time $T$ a capital not less than $\xi$.

**Lemma 54** In a c.i.-$(B, S)_T$ market with an EMM $\mathbb{P}^*$ let $\eta$ be a $(\mathcal{F}$-measurable) non-negative random variable and let $\pi = \{\eta_t := (\beta_t, \gamma_t) \mid t \in [0, T]\}$ be an $(x, \xi)$-hedging strategy where $x \in \mathbb{R}$.

Then $\{M_t^\pi \mid t \in [0, T]\}$ is a supermartingale under $\mathbb{P}^*$ and a.s. $M_T^\pi \geq 0$, $X_T^\pi \geq 0$ for each $t \in [0, T]$, particularly $x - X_0^\pi \geq 0$.

**Proof.** The integral form (22) in Lemma 52 implies immediately that $\{M_t^\pi \mid t \in [0, T]\}$ is a local martingale under $\mathbb{P}^*$. Let $\{\tau_n\}_{n=0}^\infty$ be a localizing sequence of Markov times for $M_t^\pi$.

Then with probability 1 for each $\omega \in \Omega$ there is an $n \in \mathbb{N}$ such that $\tau_n(\omega) - T$. Combining this with the fact that

$$0 \leq \xi \leq X_T^\pi = B_T M_T^\pi \quad \text{a.s.}$$

and with

$$\mathbb{E}(M_{\tau_n \wedge t}^\pi \mid \mathcal{F}_t) = M_{\tau_n \wedge t}^\pi$$

we get a.s. that $M_t^\pi$ is non-negative for each $t \in [0, T]$ and so is $X_t^\pi = B_t M_t^\pi$ for $t \in [0, T]$. Finally, by an application of Fatou’s lemma (for conditional expectation) we obtain for $s < t$ and $\delta < t$ such that $\tau_n(s) - T$

$$\mathbb{E}(M_t^\pi \mid \mathcal{F}_s) = \mathbb{E}\left(\liminf_{n \to \infty} M_{\tau_n \wedge t}^\pi \mid \mathcal{F}_s\right) \leq \liminf_{n \to \infty} \mathbb{E}\left(M_{\tau_n \wedge t}^\pi \mid \mathcal{F}_s\right) = \liminf_{n \to \infty} M_{\tau_n \wedge t}^\pi = M_s^\pi,$$

i.e., $\{M_t^\pi \mid t \in [0, T]\}$ is a supermartingale indeed. $\Box$

**Corollary 55** If there exists an equivalent martingale measure in a c.i.-$(B, S)_T$ market then the market excludes arbitrage:

there is no self-financing strategy $\pi$ with $X_0^\pi \leq 0$, $X_T^\pi > 0$ for $t \in [0, T]$ such that

$$\mathbb{P}\{X_T^\pi > 0\} > 0.$$  

**Proof.** For a self-financing strategy $\pi$ with $X_T^\pi \geq 0$ and $\mathbb{P}(X_T^\pi > 0) > 0$ Lemma 54 shows that

$$X_T^\pi \geq \frac{B_0}{B_T} \mathbb{E}^\pi X_T^\pi > 0.$$

$\square$

**Corollary 56** Let $\pi$ be a self-financing $(x, \xi)$-hedging strategy in a c.i.-$(B, S)_T$ market with EMM $\mathbb{P}^*$ where $x \in \mathbb{R}$ and $\xi$ is a non-negative $(\mathcal{F}$-measurable) random variable with $\mathbb{E}[|\xi|] < \infty$. Then

$$x \geq \frac{B_0}{B_T} \mathbb{E}^\pi \xi$$

and if additionally $\pi$ is an admissible minimal $(x, \xi)$-hedge then in inequality (27) the equality holds.

**Proof.** By Lemma 54

$$\frac{B_0}{B_T} \mathbb{E}^\pi \xi \leq \frac{B_0}{B_T} \mathbb{E}^\pi X_T^\pi = B_t \mathbb{E}^\pi M_t^\pi \leq B_t M_t^\pi - x.$$  

(28)

and the inequalities can be replaced by equalities in (28) if $\pi$ is minimal and admissible. $\square$

**Definition 57** A c.i.-$(B, S)_T$ market is said to be complete if for any $(\mathcal{F}$-measurable) random variable $\xi$ there exists a self-financing strategy $\pi$ such that $X_T^\pi = \xi \text{ a.s.}$

**Theorem 58 (Price of Contingent Claim)** Let $\xi$ be a non-negative $(\mathcal{F}$-measurable) random variable with $\mathbb{E}[|\xi|] < \infty$ in a complete c.i.-$(B, S)_T$ market with EMM $\mathbb{P}^*$. Then

$$\mathbb{C}_{\xi, \xi} = \frac{B_0}{B_T} \mathbb{E}^\pi \xi.$$  

**Proof.** By Corollary 56

$$\mathbb{C}_{\xi, \xi} = \frac{B_0}{B_T} \mathbb{E}^\pi \xi$$

holds. Moreover, by the completeness of the market there exists a self-financing strategy $\pi$ with

$$X_T^\pi = \xi \quad \text{and} \quad X_0^\pi = \frac{B_0}{B_T} \mathbb{E}^\pi \xi.$$  

Thus $\mathbb{P}(X_T^\pi, \xi) \neq \emptyset$ and $X_0^\pi \geq \mathbb{C}_{\xi, \xi}$ which completes the proof. $\square$
Appendix

In this paper we use several notions of stochastic calculus therefore we find it useful to supply here a collection of the very necessary definitions, notations and some related remarks.

Set \( \Gamma \subset \mathbb{R}^+ \), suppose that the interval \( I \) has either the form \( [0, T] \) or the form \( [0, \infty) \) and let \((\Omega, \mathcal{F})\) be a measurable space.

Then an increasing family \( \mathbb{F} = \{ \mathcal{F}_t \mid t \in \Gamma \} \) (i.e., \( \mathcal{F}_s \subset \mathcal{F}_t \), for \( s < t, s, t \in \Gamma \)) of sub-\( \sigma \)-algebras of \( \mathcal{F} \) is called filtration.

A Markov time on \((\Omega, \mathcal{F})\) is a random variable \( \tau : \Omega \to \Gamma \) such that \( \{ \tau \leq t \} \in \mathcal{F}_t \) for each \( t \in \Gamma \).

If \( \mathbb{P} \) is a probability measure on \((\Omega, \mathcal{F})\) then the quartile \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) is a filtered probability space.

Let \( \mathbb{G} = \{ G_t \mid t \in I \} \) be a filtration on \((\Omega, \mathcal{F})\). A subset of \( \Omega \times I \) is called predictable if it can be written either in the form \( \{ 0 \} \times G_t \) or in the form \( \{ s, t \} \times G \) where \( G_t \in \mathbb{G}_t \) and \( 0 \leq s < t, s, t \in I \) and \( G \in \mathbb{G} \). The predictable \( \sigma \)-algebra on \((\Omega, \mathcal{F}, \mathbb{G})\) is the \( \sigma \)-algebra generated by the predictable rectangles.

Then \( \mathbb{G} \) is said to be standard if

- \( \mathbb{G} \) is right continuous, meaning that \( G_t = \bigcap_{s \in \mathbb{G}} G_s \) for \( t \in I \), \( t \neq T \) and
- \( \mathbb{G}_0 \) contains all the null sets of \( \mathbb{P} \) in \( \mathcal{F} \).

A process on \((\Omega, \mathcal{F})\) is a collection \( X = \{ X_t \mid t \in \Gamma \} \) where \( X_t \) is \( \mathcal{F} \)-measurable for each \( t \in \Gamma \). Therefore one can consider a process as a map from \( \Omega \times I \) to \( \mathbb{R} \) such that the image of \( (\omega, t) \in \Omega \times I \) is \( X_t(\omega) \). We will sometimes use the simple notation \( X \) for a process if it is clear what the index set \( \Gamma \) would be. Taking an element \( \omega \in \Omega \) the set \( \{ X_t(\omega) \mid t \in \Gamma \} \) is called a trajectory or (sample) path of \( X \). The process \( X \) is (left, right) continuous if its trajectories are so, meaning that the functions \( X_t(\omega) : \Gamma \to \mathbb{R}, \omega \in \Omega \), are continuous. The process \( X \) is called adapted to the filtration \( \mathbb{F} \) if \( X_t \) is \( \mathcal{F}_t \)-measurable for each \( t \in \Gamma \). We say that a real valued function on \( \Gamma \) is a deterministic process on \((\Omega, \mathcal{F})\). Similarly, the process \( X \) is said to have left (right) limit if its trajectories have left (right) limit for each \( \omega \in \Omega \) and \( X_{t-} \) \( (X_{t+}) \) denotes the the left (right) limit at \( t \in \Gamma \). Furthermore, \( \Delta X_t := X_t - X_{t-} \) if \( X_{t+} \) exists. An RCLL (LCRL) process is a right (left) continuous process having left (right) limit.

Let \( M = \{ M_t \mid t \in \Gamma \} \) be an integrable process on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) by which we mean that \( \mathbb{E}\{ M_t \mid \mathcal{F}_s \} < \infty, t \in \Gamma \). Then \( M \) is called

- supermartingale if \( \mathbb{E}\{ M_t \mid \mathcal{F}_s \} \leq M_s \) for \( s < t, s, t \in \Gamma \);
- submartingale if \( \mathbb{E}\{ M_t \mid \mathcal{F}_s \} \geq M_s \) for \( s < t, s, t \in \Gamma \);
- martingale if \( M \) is both supermartingale and submartingale.
- \( L_p \)-martingale, \( p \in (0, \infty) \) if \( M \) is a martingale and \( p \)-integrable: \( \mathbb{E}\{ |M_t|^p \} < \infty \) for \( t \in \Gamma \).

Now recall the form of \( I \) and \( \mathcal{G} \). A process on \((\Omega, \mathcal{F}, \mathbb{G})\) is predictable if it is measurable with respect to the predictable \( \sigma \)-algebra (here consider the process as a map from \( \Omega \times I \) to \( \mathbb{R} \)). An adapted process \( L = \{ L_t \mid t \in \Gamma \} \) on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})\) is called local \((L_p)\)-martingale \( (p \in (0, \infty) \) if there exists an increasing sequence \( \{ \tau_n \}_{n=1}^\infty \) of Markov times, called localizing sequence, such that

\[ \tau_n \to \infty \text{ a.s. as } n \to \infty \text{ if } I = [0, \infty) \]

or

\[ \mathbb{P}(\tau_n - \tau) \to 1 \text{ as } n \to \infty \text{ if } I = [0, T] \]

and for each \( k \in \mathbb{N} \) \( \{ M_{\tau_n+k} \mid t \in I \} \) forms a \((L_p)\)-martingale.

An adapted process on \((\Omega, \mathcal{F}, \mathbb{G})\) is locally of bounded variation or called also a process of finite variation if the variation of each trajectory of the process is finite over \([0, t] \) for all \( t \in I \).

A semimartingale on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})\) is an adapted process, say, \( Y = \{ Y_t \mid t \in I \} \) which admits the representation

\[ Y_t = Z_t + A_t \text{ for each } t \in I \]

where \( Z = \{ Z_t \mid t \in I \} \) is a local martingale and \( A = \{ A_t \mid t \in I \} \) is a process of finite variation.

As it is usual in the literature, we use the brackets \( [\cdot, \cdot] \) to denote the joint variation of two processes which is also called quadratic variation if the processes are the same.

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10 Bibliographic notes

There are several articles and books on option theory and the pricing problem. First of all we mention [Bachelier 1900] and [Samuelson 65] since in these early papers the first important steps to characterize the stock price evolution and to approach the pricing problems (with the aid of stochastic calculus) were made. Then [Black and Scholes 73], [Cox, Ross and Rubinstein 79] and [Harrison and Pliska 81] have obtained their fundamental results of pricing the derivatives.

As we already mentioned Dzhaparidze and Zuijlen [Dzhaparidze and Zuijlen 90] provide a non-probabilistic approach for the same pricing problems. We have found the work of Harrison and Pliska [Harrison and Pliska 81] and the excellent expositions of Shiryayev (Shiryayev 94), [Shiryayev, Kabanov, Kramkov, Mel'nikov I. 94], [Shiryayev, Kabanov, Kramkov, Mel'nikov II. 94] the most useful for our purposes.

Further material on security markets and interesting related topics can be found in [Duffie 92] and here we also refer to the papers [Shiryayev, Kabanov, Kramkov, Mel'nikov I. 94] and [Shiryayev, Kabanov, Kramkov, Mel'nikov II. 94]. For a detailed economic discussion on derivative securities we refer to [Hull 93].

Like for many other topics of mathematics and finance, the books [Liptser and Shiryaev 89], [Rogers and Williams] and [Chung and Williams 90] provide excellent expositions of stochastic calculus with many examples and applications.

References


